PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

The Disturbance due to a Line Source in a Semi-Infinite Elastic Medium with a Single Surface Layer

Margery Newlands

Phil. Trans. R. Soc. Lond. A 1952 **245**, 213-308
doi: 10.1098/rsta.1952.0023

THE DISTURBANCE DUE TO A LINE SOURCE IN A SEMI-INFINITE ELASTIC MEDIUM WITH A SINGLE SURFACE LAYER

By MARGERY NEWLANDS, *Newnham College, Cambridge*

(*Communicated by R. Stoneley, F.R.S.—Received* 14 *December* 1951)

It has long been recognized that the simple ray theory provides only a very incomplete picture of the disturbance at a point due to a sudden localized movement in an elastic medium.

In this paper an investigation is made of the disturbance created by a cylindrical pulse (of *P*- and *S*-type) emitted from a line source in a surface layer of elastic material overlying a semi-infinite medium of different elastic constants and density.

An exact formal description of the motion is obtained in terms of a succession of pulses; the double integrals corresponding to each are evaluated by approximate methods. It is found that at a remote point (at or near the surface) there should be felt pulses corresponding to travel by each one of the minimum-time-paths predicted by the ray theory, and, in addition, a whole series of diffraction effects. Ray-path pulses are of the same type as the initial pulse, showing the same 'jerk' in the displacements (or in the rate-of-change of these); diffraction pulses are in general 'blunt', but certain of them become sharper as the surface is approached until, at the surface, they become part of a minimum-time-path disturbance.

The apparent *S*- and *Sg*-anomalies are considered in the light of these results.

At a certain range interference between pulses becomes important, and at very great range the dispersive Rayleigh wave-train becomes the dominant feature. A further study of the propagation of free Rayleigh waves shows that an infinite number of modes of vibration are possible. The degree to which each is excited and the resultant motion is determined in part II; the importance of the Airy phases is demonstrated.

The pulse representation has a natural extension to systems of any number of layers; before the corresponding interference pattern at great range can be determined it will be essential to extend our knowledge of the dispersion of free surface waves to such multilayered systems.

CONTENTS

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

PART I. THE PULSE REPRESENTATION OF THE DISTURBANCE DUE TO THE LINE SOURCE

1. Introduction

An unbounded elastic medium possessing rigidity and compressibility can propagate body-waves of two types, compressional (*P*-waves) and distortional (*S*-waves), the latter denoted by *SV* when polarized in the vertical plane through the direction of propagation and by *SH* when polarized in the horizontal direction.

A plane wave of *P*- or *S*-type undergoes reflexion at an interface between different media according to stationary-time path laws analogous to those of geometrical optics. Similarly, there arises the idea of a critical angle of incidence, but the phenomenon is more complicated since, supposing the interface to be horizontal, both *P*- and *SV*-waves are ordinarily broken up into four waves, each into reflected and refracted *P*- and *SV*-waves; an *SH*-wave, however, gives only reflected and refracted *SH*-waves.

When the medium is bounded by an infinite plane surface it can transmit Rayleigh waves in which the motion is confined to the neighbourhood of the surface and is partly in the direction of propagation and partly normal to the surface (Rayleigh 1885). If, in addition, the system possesses a uniform surface layer of differing properties, then, subject to a certain condition on these properties, a second type of surface motion is possible in which the displacement is purely horizontal and normal to the direction of propagation (Love 1911). These Love waves are dispersive. Generalized Rayleigh waves in systems with one or more surface layers also show a dependence of phase-velocity on wave-length.

So far no reference has been made to the actual generation of these various possible wave-types. The practical interest of the study of elastic waves lies in the interpretation of records of earthquakes or of artificial explosions to deduce the nature of the materials which lie below the earth's surface. The above theoretical considerations assume continuous propagation. In practice, the disturbance at the energy source is a pulse of short duration and at best may be described by a superposition of waves of all frequencies spreading cylindrically or spherically from the source. Nevertheless, the interpretation of records is based essentially on the simple ideas of reflexion and refraction of *P*- and *S*-waves together with the notion of 'rays' along which the energy is propagated and the assumption that, in general, the record will exhibit arrivals corresponding to the surface Rayleigh and Love waves. In fact, the seismogram presents a highly complex pattern which cannot be explained on a simple ray theory. Comparison of earthquake records reveals marked inconsistencies, considerable scattering of readings, particularly of the '*S*-phase' (seismological notation), without any convincing concentration of frequency (Jeffreys 1946, p. 61); moreover, the *S*-phase and the corresponding *P*-phase are pulses which appear to have been refracted along the interface between the granitic and ultra-basic rock and up again to the surface, and on a ray theory their associated energy should be negligible; yet, the refracted pulses are prominent and, hence, useful features of earthquake and experimental records. There are also characteristics of the 'surface wave', as observed at great distances, which require more detailed explanation than just the existence of a stationary value of group velocity giving a maximum amplitude, in particular, the long trains of regular waves observed to follow the main disturbance.

At least part of the explanation is to be sought in the diffraction effect due to the curvature of the wave fronts of which the simple ray theory does not take account.

Lamb (1904) first considered the generation of a disturbance in a semi-infinite medium by the application of a vertical or horizontal impulse along a line in the surface. After intricate analysis he showed that there should in fact be felt, in succession, just a *P*-pulse, an *S*-pulse and a Rayleigh pulse, the two former abrupt like the initial pulse and the latter a 'blunt pulse' beginning slowly, rising to a maximum and dying away. Lamb indicated the procedure when the source lay below the surface and the problem was first attacked by Nakano (1925). His methods were somewhat laborious, and there was a disturbing inconsistency between the results obtained for an initial harmonic vibration and generalization to a pulse; but it seemed probable that in addition to the *P*-, *S*- and Rayleigh pulses should be felt, for initial *P*- and *S*-pulses respectively, what he termed a 'surface *S*-pulse' and a 'surface *P*-pulse'.

Jeffreys (1926*a*) considered the disturbance due to a spherical explosion in the upper of two superposed layers. He simplified the problem by neglecting rigidity; by using the Bromwich expansion method (Bromwich 1916) he was able to resolve the disturbance into an infinite series of pulses each expressed by a contour integral. He showed that there should be felt the expected direct and reflected waves, and in addition, contrary to the predictions of the 'ray theory', refracted waves of finite amplitude.

Later (Jeffreys 1931) he investigated the formation of Love waves by a 'quasi-symmetrical pulse of *SH*-type generated by the sudden application (or removal) of a rotational stress over a sphere within the upper layer of the crust'. By the same treatment and subsequent approximation the motion was resolved into a series of pulses, certain of which, Jeffreys concluded, must be equivalent to the system of Love waves without nodal planes.

Muskat (1933) considered the reflexion and refraction of waves at an interface between two elastic media of great depth and extent and obtained pulses analogous to the surface *S*-pulse and surface *P*-pulse of Nakano. Pekeris (1948) and Press, Ewing & Tolstoy (1950) investigated the disturbance due to a source in a liquid layer on an infinite depth of fluid and solid respectively, but they neglected certain branch-line integrals and studied only the surface wave part of the motion.

The problem of the disturbance due to a line source of *P*- and *S*-waves below the surface of a homogeneous semi-infinite medium—Nakano's problem—was reconsidered by Lapwood (1949) and lucidly presented using the Sommerfeld method of analysis; Nakano's inconsistency was traced to a neglected branch-line integral and the reality of the 'surface *S*'- and 'surface *P*'- pulses established. This work should throw some light on the anomalies of near-earthquake records, but, as Lapwood writes, 'the hypothesis of a homogeneous semi-infinite solid means that our work cannot account for phenomena which are due to stratification... or a variation of velocity with depth'. Thus the corresponding analysis of the problem of the cylindrical pulse in a stratified medium is likely to be of considerable seismological interest.

The following discussion treats the case of a semi-infinite medium with a single surface layer; it is believed that the results will indicate fairly clearly what added complexity may be expected in a multi-layered medium and what would be the effect of a deep-lying source.

2. General equations of motion, and boundary conditions

Let $z = 0$ be the surface of a semi-infinite medium in which a layer of uniform depth H, density ρ_1 and elastic constants λ_1, μ_1, overlies a medium of very great depth and of density ρ_2 and elastic constants λ_2, μ_2. It is required to investigate the disturbance at a general point G within the surface layer due to a cylindrical pulse emitted from a line source L parallel to the interface and at a depth h below the surface. It is therefore convenient to use right-handed axes Ox, Oy, Oz with origin in the surface, so that the line source is given by $x = 0$, $z = h$, and the interface by $z = H$ (figure 1). The motion will be two-dimensional and independent of y.

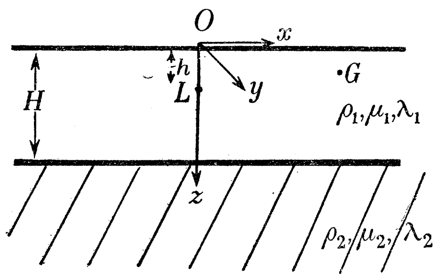

Figure 1

The equations of small motion of an elastic solid in the absence of body forces may be written vectorially as

$$(\lambda+\mu)\,\mathrm{grad}\,\Delta+\mu\nabla^2\mathbf{v} = \rho\,\frac{\partial^2\Delta}{\partial t^2}, \tag{2·1}$$

where $\mathbf{v}$ is the displacement (u, v, w) and $\Delta = \mathrm{div}\,\mathbf{v}$ (Love 1906). Bromwich (1898) showed that in a problem of this type the effect of gravity is negligible.

In general, the displacement at a point may be expressed in terms of scalar and vector potentials ϕ and ψ by the relation

$$\mathbf{v} = \mathrm{grad}\,\phi+\mathrm{curl}\,\boldsymbol{\psi}, \tag{2·2}$$

whence the equations of motion take the simple forms

$$(\lambda+2\mu)\,\nabla^2\phi = \rho\,\frac{\partial^2\phi}{\partial t^2}, \tag{2·3}$$

$$\mu\nabla^2\boldsymbol{\psi} = \rho\,\frac{\partial^2\boldsymbol{\psi}}{\partial t^2}. \tag{2·4}$$

Returning to the two-dimensional problem in which the displacements u, w are functions of x, z only, we have

$$u = \frac{\partial\phi}{\partial x}+\frac{\partial\psi}{\partial z}, \quad w = \frac{\partial\phi}{\partial z}-\frac{\partial\psi}{\partial x}, \tag{2·5}$$

where ψ is strictly the second component of a vector defined by (2·2) and ϕ and ψ satisfy

$$\nabla^2\phi = \frac{1}{\alpha^2}\frac{\partial^2\phi}{\partial t^2}, \tag{2·6}$$

$$\nabla^2\psi = \frac{1}{\beta^2}\frac{\partial^2\psi}{\partial t^2}, \tag{2·7}$$

$$\left.\begin{aligned} \nabla^2 &\equiv \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}, \\ \alpha &= \surd(\lambda+2\mu/\rho) \quad \text{(velocity of propagation of } P\text{-waves)}, \\ \beta &= \surd(\mu/\rho) \quad \text{(velocity of propagation of } S\text{-waves)}. \end{aligned}\right\}$$

The normal and tangential stresses at any point on an area perpendicular to the z-axis are $\widehat{zz}$ and $\widehat{xz}$ respectively, where

$$\widehat{zz} = \lambda\left(\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}\right)+2\mu\frac{\partial w}{\partial z}$$

$$\equiv (\lambda+2\mu)\,\nabla^2\phi-2\mu\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\psi}{\partial x\,\partial z}\right), \tag{2·8}$$

and

$$\widehat{xz} = \mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)$$

$$\equiv \mu\nabla^2\psi-2\mu\left(\frac{\partial^2\psi}{\partial x^2}-\frac{\partial^2\phi}{\partial x\,\partial z}\right). \tag{2·9}$$

3. The disturbance created by an initial P-pulse

Jeffreys (1931) has suggested that for a wide class of earthquakes it is probably valid to approximate to the original disturbance by a pulse varying in time like a simple unit Heaviside function

$$H(t) = \frac{1}{2\pi \mathrm{i}}\int_\Omega \mathrm{e}^{\mathrm{i}\omega t}\frac{\mathrm{d}\omega}{\omega}\begin{cases}=0 & (t<0),\\ =1 & (t>0),\end{cases} \tag{3·1}$$

where Ω is the line parallel to the real axis in the ω-plane from $-\infty-\mathrm{i}c$ to $+\infty-\mathrm{i}c$, or any equivalent contour. We may consider separately the compressional and distortional parts of the initial disturbance. The solution of the wave equation for ϕ which varies as $\mathrm{e}^{\mathrm{i}\omega t}$ and represents a compressional disturbance travelling out cylindrically from the line source is

$$\phi_0 = \pi\mathrm{i}\,\mathrm{Hi}_0(\varpi\kappa_{\alpha_1})\,\mathrm{e}^{\mathrm{i}\omega t}, \tag{3·2}$$

where

$$\kappa_{\alpha_1} = \omega/\alpha_1, \tag{3·3}$$

(similarly we shall define κ_{β_1}, κ_{α_2}, κ_{β_2})

$$\varpi^2 = x^2+(h-z)^2, \tag{3·4}$$

and Hi_0 is the Hankel function of the second type and zero order (Jeffreys & Jeffreys, 1946, p. 544). When $|\varpi\kappa_{\alpha_1}|$ is large,

$$\mathrm{Hi}_0(\varpi\kappa_{\alpha_1}) \sim \sqrt{\frac{2\mathrm{i}}{\pi\varpi\kappa_{\alpha_1}}}\exp\{-\mathrm{i}\varpi\kappa_{\alpha_1}\}, \tag{3·5}$$

and it is seen that ϕ_0 does in fact represent a wave travelling out from the source with velocity α_1. Also $\mathrm{Hi}_0(\varpi\kappa_{\alpha_1})\to 0$ as $|\varpi\kappa_{\alpha_1}|\to\infty$, providing $\mathscr{I}(\varpi\kappa_{\alpha_1})\leqslant 0$, consistent with our definition of ω on the Ω contour. The factor $\pi\mathrm{i}$ is introduced for algebraic convenience.

By the principle of superposition it follows that when the time variation of the source is not as $\mathrm{e}^{\mathrm{i}\omega t}$ but as $H(t)$, the corresponding 'total' displacement potential is given by

$$\Phi_0 = \frac{1}{2}\int_\Omega \mathrm{Hi}_0(\varpi\kappa_{\alpha_1})\,\mathrm{e}^{\mathrm{i}\omega t}\frac{\mathrm{d}\omega}{\omega}. \tag{3·6}$$

In order to separate the x, z occurring here in the combination ϖ, we use an extension by Lapwood (1949, p. 67) of a result of Lamb (1904, p. 4) and write

$$\mathrm{Hi}_0(\varpi\kappa_{\alpha_1}) = -\frac{2}{\pi\mathrm{i}}\int_0^\infty \exp\{\mp z\lambda_{\alpha_1}\}\cos\zeta x\frac{\mathrm{d}\zeta}{\lambda_{\alpha_1}}\quad (z\gtrless 0), \tag{3·7}$$

where $$\varpi^2 = x^2+z^2$$

and $$\lambda_{\alpha_1} = \surd(\zeta^2-\kappa^2_{\alpha_1}), \quad \mathscr{R}(\lambda_{\alpha_1})>0. \tag{3·8}$$

Later we shall introduce λ_{β_1}, λ_{α_2}, λ_{β_2}, defined likewise.

Thus, returning to the problem of a source at depth h in the surface layer, we may describe an initial P-pulse by

$$\Phi_0 = \frac{1}{2\pi \mathrm{i}}\int_\Omega \frac{\mathrm{d}\omega}{\omega}\int_0^\infty (-2)\exp\{\mp(h-z)\,\lambda_{\alpha_1}\}\cos\zeta x\frac{\mathrm{d}\zeta}{\lambda_{\alpha_1}}\,\mathrm{e}^{\mathrm{i}\omega t} \quad (0\leqslant z\leqslant h;\ h\leqslant z\leqslant H), \tag{3·9}$$

where corresponding to a single value of ω we have

$$\phi_0 = -2\int_0^\infty \exp\{\mp(h-z)\,\lambda_{\alpha_1}\}\cos\zeta x\frac{\mathrm{d}\zeta}{\lambda_{\alpha_1}}\,\mathrm{e}^{\mathrm{i}\omega t} \quad (0\leqslant z\leqslant h;\ h\leqslant z\leqslant H). \tag{3·10}$$

We must now satisfy the boundary conditions, that is, the vanishing of the normal and tangential stresses at the surface and the continuity of displacements and stresses at the interface. These may be written

$$u\equiv\frac{\partial\phi}{\partial x}+\frac{\partial\psi}{\partial z} \quad \text{continuous at} \quad z=h, \tag{3·11}$$

$$w\equiv\frac{\partial\phi}{\partial z}-\frac{\partial\psi}{\partial x} \quad \text{continuous at} \quad z=h, \tag{3·12}$$

$$\widehat{xz}\equiv\mu(2\phi_{xz}+\psi_{zz}-\psi_{xx}) \quad \text{continuous at} \quad z=h \tag{3·13}$$

$$\text{and vanishes at } z=0, \tag{3·14}$$

$$\widehat{zz}=(\lambda+2\mu)\,\nabla^2\phi-2\mu(\phi_{xx}+\psi_{xz}) \quad \text{continuous at} \quad z=h \tag{3·15}$$

$$\text{and vanishes at } z=0, \tag{3·16}$$

where ϕ_{xz} denotes $\dfrac{\partial^2\phi}{\partial x\partial z}$, etc. The normal stress at the surface is nullified by an equal and opposite source at the image line $x=0$, $z=-h$, given by

$$\phi_r = 2\int_0^\infty \exp\{-(h+z)\,\lambda_{\alpha_1}\}\cos\zeta x\frac{\mathrm{d}\zeta}{\lambda_{\alpha_1}}\,\mathrm{e}^{\mathrm{i}\omega t} \quad (z\geqslant -h). \tag{3·17}$$

Combining ϕ_0 and ϕ_r we have

$$\phi_{0r}\equiv\phi_0+\phi_r = -4\int_0^\infty \exp\{-h\lambda_{\alpha_1}\}\sinh(z\lambda_{\alpha_1})\cos\zeta x\frac{\mathrm{d}\zeta}{\lambda_{\alpha_1}}\,\mathrm{e}^{\mathrm{i}\omega t} \quad (0\leqslant z\leqslant h) \tag{3·18}$$

$$= -4\int_0^\infty \exp\{-z\lambda_{\alpha_1}\}\sinh(h\lambda_{\alpha_1})\cos\zeta x\frac{\mathrm{d}\zeta}{\lambda_{\alpha_1}}\,\mathrm{e}^{\mathrm{i}\omega t} \quad (h\leqslant z\leqslant H). \tag{3·19}$$

In order to satisfy the boundary conditions completely, we add potentials ϕ and ψ to ϕ_{0r} in the surface layer and attempt to describe the motion in the lower medium by suitable ϕ and ψ. The equations (3·11) to (3·16) and the form of ϕ_{0r} suggest solutions of the type

upper layer
$$\phi = 4\int_0^\infty (A\exp\{-(z-H)\,\lambda_{\alpha_1}\}+B\exp\{(z-H)\,\lambda_{\alpha_1}\})\cos\zeta x\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\zeta, \tag{3·20}$$

$$\psi = 4\int_0^\infty (C\exp\{-(z-H)\,\lambda_{\beta_1}\}+D\exp\{(z-H)\,\lambda_{\beta_1}\})\sin\zeta x\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\zeta; \tag{3·21}$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

$$\text{lower medium}\begin{cases}\overline{\phi} = 4\int_0^\infty R\exp\{-(z-H)\,\lambda_{\alpha_2}\}\cos\zeta x\,\mathrm{d}\zeta\,\mathrm{e}^{\mathrm{i}\omega t}, & (3\cdot22)\\ \overline{\psi} = 4\int_0^\infty Q\exp\{-(z-H)\,\lambda_{\beta_2}\}\sin\zeta x\,\mathrm{d}\zeta\,\mathrm{e}^{\mathrm{i}\omega t}. & (3\cdot23)\end{cases}$$

Substituting (3·20) to (3·23) and (3·18) and (3·19) in the boundary conditions (3·11) to (3·17), we obtain six integrals which must vanish for all values of x and so the integrands must be zero. This leads to six equations in A, B, C, D, R, Q:

$$-\zeta A-\zeta B-\lambda_{\beta_1}C+\lambda_{\beta_1}D+\zeta\exp\{-H\lambda_{\alpha_1}\}\sinh(h\lambda_{\alpha_1})/\lambda_{\alpha_1} = -\zeta R-\lambda_{\beta_2}Q, \quad (3\cdot24)$$

$$-\lambda_{\alpha_1}A+\lambda_{\alpha_1}B-\zeta C-\zeta D+\exp\{-H\lambda_{\alpha_1}\}\sinh(h\lambda_{\alpha_1}) = -\lambda_{\alpha_2}R-\zeta Q, \quad (3\cdot25)$$

$$2\zeta\lambda_{\alpha_1}A-2\zeta\lambda_{\alpha_1}B+(2\zeta^2-\kappa_{\beta_1}^2)\,C+(2\zeta^2-\kappa_{\beta_1}^2)\,D-2\zeta\exp\{-H\lambda_{\alpha_1}\}\sinh(h\lambda_{\alpha_1}) = 2\zeta\lambda_{\alpha_2}(\mu_2/\mu_1)\,R+(2\zeta^2-\kappa_{\beta_2}^2)\,(\mu_2/\mu_1)\,Q, \quad (3\cdot26)$$

$$(2\zeta^2-\kappa_{\beta_1}^2)\,A+(2\zeta^2-\kappa_{\beta_1}^2)\,B+2\zeta\lambda_{\beta_1}C-2\zeta\lambda_{\beta_1}D-(2\zeta^2-\kappa_{\beta_1}^2)\exp\{-H\lambda_{\alpha_1}\}\sinh(h\lambda_{\alpha_1})/\lambda_{\alpha_1} = (2\zeta^2-\kappa_{\beta_2}^2)\,(\mu_2/\mu_1)\,R+2\zeta\lambda_{\beta_2}(\mu_2/\mu_1)\,Q, \quad (3\cdot27)$$

$$2\zeta\lambda_{\alpha_1}\exp\{H\lambda_{\alpha_1}\}\,A-2\zeta\lambda_{\alpha_1}\exp\{-H\lambda_{\alpha_1}\}\,B+(2\zeta^2-\kappa_{\beta_1}^2)\exp\{H\lambda_{\beta_1}\}\,C+(2\zeta^2-\kappa_{\beta_1}^2)\exp\{-H\lambda_{\beta_1}\}\,D+2\zeta\exp\{-h\lambda_{\alpha_1}\}=0, \quad (3\cdot28)$$

$$(2\zeta^2-\kappa_{\beta_1}^2)\exp\{H\lambda_{\alpha_1}\}\,A+(2\zeta^2-\kappa_{\beta_1}^2)\exp\{-H\lambda_{\alpha_1}\}\,B+2\zeta\lambda_{\beta_1}\exp\{H\lambda_{\beta_1}\}\,C-2\zeta\lambda_{\beta_1}\exp\{-H\lambda_{\beta_1}\}\,D=0, \quad (3\cdot29)$$

from which each of $A, B, \ldots$ is expressible as the quotient of two 6×6 determinants. The common denominator we shall denote by Δ_p and the other six determinants by $\Delta_A, \Delta_B, \ldots,$ so that

$$A = \Delta_A/\Delta_p, \quad \text{etc.}$$

It is convenient to write $\Delta_p, \Delta_A, \ldots$ as follows:

$$\Delta_p = \exp\{H(\lambda_{\alpha_1}+\lambda_{\beta_1})\}\,S'\left(1+\frac{T'}{S'}\exp\{-2H\lambda_{\beta_1}\}+\frac{V'+Y'}{S'}\exp\{-H(\lambda_{\alpha_1}+\lambda_{\beta_1})\}+\frac{W'}{S'}\exp\{-2H\lambda_{\alpha_1}\}+\frac{U'}{S'}\exp\{-2H(\lambda_{\alpha_1}+\lambda_{\beta_1})\}\right), \quad (3\cdot30)$$

where

$$\left.\begin{aligned} S' &= FS,\\ T' &= -\overline{F}T,\\ V' &= 4\zeta\lambda_{\alpha_1}(2\zeta^2-\kappa_{\beta_1}^2)\,V,\\ Y' &= -4\zeta\lambda_{\beta_1}(2\zeta^2-\kappa_{\beta_1}^2)\,Y,\\ W' &= -\overline{F}W,\\ U' &= FU;\end{aligned}\right\} \quad (3\cdot31)$$

and

$$\left.\begin{aligned} F(\zeta) &= (2\zeta^2-\kappa_{\beta_1}^2)^2-4\zeta^2\lambda_{\alpha_1}\lambda_{\beta_1},\\ \overline{F}(\zeta) &= (2\zeta^2-\kappa_{\beta_1}^2)^2+4\zeta^2\lambda_{\alpha_1}\lambda_{\beta_1};\end{aligned}\right\} \quad (3\cdot32)$$

$$\left.\begin{aligned}
S &= \begin{vmatrix} -\zeta & \lambda_{\beta_1} & -\zeta & -\lambda_{\beta_2} \\ \lambda_{\alpha_1} & -\zeta & -\lambda_{\alpha_2} & -\zeta \\ -2\zeta\lambda_{\alpha_1} & (2\zeta^2-\kappa^2_{\beta_1}) & 2\zeta\lambda_{\alpha_2}\mu_2/\mu_1 & (2\zeta^2-\kappa^2_{\beta_2})\,\mu_2/\mu_1 \\ (2\zeta^2-\kappa^2_{\beta_1}) & -2\zeta\lambda_{\beta_1} & (2\zeta^2-\kappa^2_{\beta_2})\,\mu_2/\mu_1 & 2\zeta\lambda_{\beta_2}\mu_2/\mu_1 \end{vmatrix}; \\
T &= \begin{vmatrix} -\zeta & -\lambda_{\beta_1} & \cdot & \cdot \\ \lambda_{\alpha_1} & -\zeta & \cdot & \cdot \\ -2\zeta\lambda_{\alpha_1} & (2\zeta^2-\kappa^2_{\beta_1}) & \cdot & \cdot \\ (2\zeta^2-\kappa^2_{\beta_1}) & 2\zeta\lambda_{\beta_1} & \cdot & \cdot \end{vmatrix}; \quad
V = \begin{vmatrix} -\lambda_{\beta_1} & \lambda_{\beta_1} & \cdot & \cdot \\ -\zeta & -\zeta & \cdot & \cdot \\ (2\zeta^2-\kappa^2_{\beta_1}) & (2\zeta^2-\kappa^2_{\beta_1}) & \cdot & \cdot \\ 2\zeta\lambda_{\beta_1} & -2\zeta\lambda_{\beta_1} & \cdot & \cdot \end{vmatrix}; \\
Y &= \begin{vmatrix} -\zeta & -\zeta & \cdot & \cdot \\ -\lambda_{\alpha_1} & \lambda_{\alpha_1} & \cdot & \cdot \\ 2\zeta\lambda_{\alpha_1} & -2\zeta\lambda_{\alpha_1} & \cdot & \cdot \\ (2\zeta^2-\kappa^2_{\beta_1}) & (2\zeta^2-\kappa^2_{\beta_1}) & \cdot & \cdot \end{vmatrix}; \quad
W = \begin{vmatrix} -\zeta & \lambda_{\beta_1} & \cdot & \cdot \\ -\lambda_{\alpha_1} & -\zeta & \cdot & \cdot \\ 2\zeta\lambda_{\alpha_1} & (2\zeta^2-\kappa^2_{\beta_1}) & \cdot & \cdot \\ (2\zeta^2-\kappa^2_{\beta_1}) & -2\zeta\lambda_{\beta_1} & \cdot & \cdot \end{vmatrix}; \\
U &= \begin{vmatrix} -\zeta & -\lambda_{\beta_1} & \cdot & \cdot \\ -\lambda_{\alpha_1} & -\zeta & \cdot & \cdot \\ 2\zeta\lambda_{\alpha_1} & (2\zeta^2-\kappa^2_{\beta_1}) & \cdot & \cdot \\ (2\zeta^2-\kappa^2_{\beta_1}) & 2\zeta\lambda_{\beta_1} & \cdot & \cdot \end{vmatrix}.
\end{aligned}\right\} \tag{3·33}$$

The third and fourth columns of $T, V, \ldots$ are identical with those of S.

We may remark here that S, T, W and U are all even in ζ and V and Y odd. Also that

$$\lambda_{\alpha_1} V = -\lambda_{\beta_1} Y, \tag{3·34}$$

$$\begin{aligned}
4\lambda_{\alpha_1}\Delta_A \exp\{-(z-H)\,\lambda_{\alpha_1}\} &= 4\exp\{-z\lambda_{\alpha_1}+H(\lambda_{\alpha_1}+\lambda_{\beta_1})\}\,[\exp\{-h\lambda_{\alpha_1}\}\,4\zeta^2\lambda_{\alpha_1}\lambda_{\beta_1}S \\
&\quad +\exp\{-h\lambda_{\alpha_1}-2H\lambda_{\beta_1}\}\,4\zeta^2\lambda_{\alpha_1}\lambda_{\beta_1}T \\
&\quad -\exp\{-h\lambda_{\alpha_1}-H(\lambda_{\alpha_1}+\lambda_{\beta_1})\}\,2\zeta\lambda_{\alpha_1}(2\zeta^2-\kappa^2_{\beta_1})\,V \\
&\quad \mp\tfrac{1}{2}\exp\{\pm h\lambda_{\alpha_1}-H(\lambda_{\alpha_1}+\lambda_{\beta_1})\}\,4\zeta\lambda_{\beta_1}(2\zeta^2-\kappa^2_{\beta_1})\,Y \\
&\quad \mp\tfrac{1}{2}\exp\{\pm h\lambda_{\alpha_1}-2H\lambda_{\alpha_1}\}\,\overline{F}W \\
&\quad \pm\tfrac{1}{2}\exp\{\pm h\lambda_{\alpha_1}-2H(\lambda_{\alpha_1}+\lambda_{\beta_1})\}\,FU],
\end{aligned} \tag{3·35}$$

$$\begin{aligned}
4\lambda_{\alpha_1}\Delta_B \exp\{(z-H)\,\lambda_{\alpha_1}\} &= 4\exp\{z\lambda_{\alpha_1}+H(\lambda_{\alpha_1}+\lambda_{\beta_1})\}\quad [\exp\{-h\lambda_{\alpha_1}-H(\lambda_{\alpha_1}+\lambda_{\beta_1})\}\,2\zeta\lambda_{\alpha_1}(2\zeta^2-\kappa^2_{\beta_1})\,V \\
&\quad -\exp\{-h\lambda_{\alpha_1}-2H\lambda_{\alpha_1}\}\,4\zeta^2\lambda_{\alpha_1}\lambda_{\beta_1}W \\
&\quad \pm\tfrac{1}{2}\exp\{\pm h\lambda_{\alpha_1}-2H\lambda_{\alpha_1}\}\,FW \\
&\quad \mp\tfrac{1}{2}\exp\{\pm h\lambda_{\alpha_1}-2H(\lambda_{\alpha_1}+\lambda_{\beta_1})\}\,\overline{F}U \\
&\quad -\exp\{-h\lambda_{\alpha_1}-2H(\lambda_{\alpha_1}+\lambda_{\beta_1})\}\,4\zeta^2\lambda_{\alpha_1}\lambda_{\beta_1}U],
\end{aligned} \tag{3·36}$$

$$\begin{aligned}
4\lambda_{\alpha_1}\Delta_C \exp\{-(z-H)\,\lambda_{\beta_1}\} &= 4\exp\{-z\lambda_{\beta_1}+H(\lambda_{\alpha_1}+\lambda_{\beta_1})\}\,[-\exp\{-h\lambda_{\alpha_1}\}\,2\zeta\lambda_{\alpha_1}(2\zeta^2-\kappa^2_{\beta_1})\,S \\
&\quad +\exp\{-h\lambda_{\alpha_1}-H(\lambda_{\alpha_1}+\lambda_{\beta_1})\}\,4\zeta^2\lambda_{\alpha_1}\lambda_{\beta_1}Y \\
&\quad \pm\tfrac{1}{2}\exp\{\pm h\lambda_{\alpha_1}-H(\lambda_{\alpha_1}+\lambda_{\beta_1})\}\,\overline{F}Y \\
&\quad +\exp\{-h\lambda_{\alpha_1}-2H\lambda_{\alpha_1}\}\,2\zeta\lambda_{\alpha_1}(2\zeta^2-\kappa^2_{\beta_1})\,W \\
&\quad \pm\tfrac{1}{2}\exp\{\pm h\lambda_{\alpha_1}-2H\lambda_{\alpha_1}\}\,4\zeta\lambda_{\alpha_1}(2\zeta^2-\kappa^2_{\beta_1})\,W],
\end{aligned} \tag{3·37}$$

$$
\begin{aligned}
&4\lambda_{\alpha_1}\Delta_D \exp\{(z-H)\lambda_{\beta_1}\} \\
&\quad = 4\exp\{z\lambda_{\beta_1}+H(\lambda_{\alpha_1}+\lambda_{\beta_1})\} \quad [\exp\{-h\lambda_{\alpha_1}-2H\lambda_{\beta_1}\}\,2\zeta\lambda_{\alpha_1}(2\zeta^2-\kappa^2_{\beta_1})\,T \\
&\qquad +\exp\{-h\lambda_{\alpha_1}-H(\lambda_{\alpha_1}+\lambda_{\beta_1})\}\,4\zeta^2\lambda_{\alpha_1}\lambda_{\beta_1}\,Y \\
&\qquad \mp\tfrac{1}{2}\exp\{\pm h\lambda_{\alpha_1}-H(\lambda_{\alpha_1}+\lambda_{\beta_1})\}\,FY \\
&\qquad -\exp\{-h\lambda_{\alpha_1}-2H(\lambda_{\alpha_1}+\lambda_{\beta_1})\}\,2\zeta\lambda_{\alpha_1}(2\zeta^2-\kappa^2_{\beta_1})\,U \\
&\qquad \mp\tfrac{1}{2}\exp\{\pm h\lambda_{\alpha_1}-2H(\lambda_{\alpha_1}+\lambda_{\beta_1})\}\,4\zeta\lambda_{\alpha_1}(2\zeta^2-\kappa^2_{\beta_1})\,U].
\end{aligned} \tag{3·38}
$$

For the present attention will be confined to the motion in the upper layer, since we are primarily interested in surface disturbances. We may now write the formal solution for this layer in the case of an initial P-pulse as

$$
\begin{aligned}
\Phi_p &= \Phi_{0r}+\Phi \\
&= \frac{1}{2\pi \mathrm{i}}\int_\Omega \frac{\mathrm{d}\omega}{\omega}\int_0^\infty (-4)\exp\{-h\lambda_{\alpha_1}\}\sinh(z\lambda_{\alpha_1})\cos\zeta x\,\frac{\mathrm{d}\zeta}{\lambda_{\alpha_1}}\,\mathrm{e}^{\mathrm{i}\omega t} \\
&\quad +\frac{1}{2\pi \mathrm{i}}\int_\Omega \frac{\mathrm{d}\omega}{\omega}\int_0^\infty \left(4\frac{\Delta_A}{\Delta_p}\exp\{-(z-H)\lambda_{\alpha_1}\}+4\frac{\Delta_B}{\Delta_p}\exp\{(z-H)\lambda_{\alpha_1}\}\right)\cos\zeta x\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\zeta \quad (0\leqslant z\leqslant h),
\end{aligned} \tag{3·39}
$$

or

$$
\begin{aligned}
&= \frac{1}{2\pi \mathrm{i}}\int_\Omega \frac{\mathrm{d}\omega}{\omega}\int_0^\infty (-4)\exp\{-z\lambda_{\alpha_1}\}\sinh(h\lambda_{\alpha_1})\cos\zeta x\,\frac{\mathrm{d}\zeta}{\lambda_{\alpha_1}}\,\mathrm{e}^{\mathrm{i}\omega t} \\
&\quad +\frac{1}{2\pi \mathrm{i}}\int_\Omega \frac{\mathrm{d}\omega}{\omega}\int_0^\infty \left(4\frac{\Delta_A}{\Delta_p}\exp\{-(z-H)\lambda_{\alpha_1}\}+4\frac{\Delta_B}{\Delta_p}\exp\{(z-H)\lambda_{\alpha_1}\}\right)\cos\zeta x\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\zeta \quad (h\leqslant z\leqslant H).
\end{aligned} \tag{3·40}
$$

$$
\Psi_p = \Psi = \frac{1}{2\pi \mathrm{i}}\int_\Omega \frac{\mathrm{d}\omega}{\omega}\int_0^\infty \left(4\frac{\Delta_C}{\Delta_p}\exp\{-(z-H)\lambda_{\beta_1}\}+4\frac{\Delta_D}{\Delta_p}\exp\{(z-H)\lambda_{\beta_1}\}\right)\cos\zeta x\,\mathrm{d}\zeta\,\mathrm{e}^{\mathrm{i}\omega t} \quad (0\leqslant z\leqslant H). \tag{3·41}
$$

4. Formal solution for an initial S-pulse

When the original disturbance is an S-pulse then similarly it may be described by Ψ_0, where

$$
\Psi_0 = \frac{1}{2\pi \mathrm{i}}\int_\Omega \frac{\mathrm{d}\omega}{\omega}\int_0^\infty (-2)\exp\{\mp(h-z)\lambda_{\beta_1}\}\cos\zeta x\,\frac{\mathrm{d}\zeta}{\lambda_{\beta_1}}\,\mathrm{e}^{\mathrm{i}\omega t} \quad (0<z\leqslant h;\ H>z\geqslant h). \tag{4·1}
$$

The appropriate image pulse is

$$
\Psi_r = \frac{1}{2\pi \mathrm{i}}\int_\Omega \frac{\mathrm{d}\omega}{\omega}\int_0^\infty 2\exp\{-(h+z)\lambda_{\beta_1}\}\cos\zeta x\,\frac{\mathrm{d}\zeta}{\lambda_{\beta_1}}\,\mathrm{e}^{\mathrm{i}\omega t} \quad (H>z\geqslant 0). \tag{4·2}
$$

Thus

$$
\Psi_{0r} = \frac{1}{2\pi \mathrm{i}}\int_\Omega \frac{\mathrm{d}\omega}{\omega}\int_0^\infty (-4)\exp\{-h\lambda_{\beta_1}\}\sinh(z\lambda_{\beta_1})\cos\zeta x\,\frac{\mathrm{d}\zeta}{\lambda_{\beta_1}}\,\mathrm{e}^{\mathrm{i}\omega t} \quad (0<z\leqslant h), \tag{4·3}
$$

$$
= \frac{1}{2\pi \mathrm{i}}\int_\Omega \frac{\mathrm{d}\omega}{\omega}\int_0^\infty (-4)\exp\{-z\lambda_{\beta_1}\}\sinh(h\lambda_{\beta_1})\cos\zeta x\,\frac{\mathrm{d}\zeta}{\lambda_{\beta_1}}\,\mathrm{e}^{\mathrm{i}\omega t} \quad (H>z\geqslant h), \tag{4·4}
$$

and the boundary conditions are satisfied by adding potentials ϕ and ψ as before so that

upper layer

$$
\Phi_s = \frac{1}{2\pi \mathrm{i}}\int_\Omega \frac{\mathrm{d}\omega}{\omega}\int_0^\infty \left(4\frac{\Delta'_A}{\Delta_s}\exp\{-(z-H)\lambda_{\alpha_1}\}\,4\frac{\Delta'_B}{\Delta_s}\exp\{(z-H)\lambda_{\alpha_1}\}\right)\sin\zeta x\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\zeta, \tag{4·5}
$$

$$
\Psi_s = \Psi_{0r}+\frac{1}{2\pi \mathrm{i}}\int_\Omega \frac{\mathrm{d}\omega}{\omega}\int_0^\infty \left(4\frac{\Delta'_C}{\Delta_s}\exp\{-(z-H)\lambda_{\beta_1}\}+4\frac{\Delta'_D}{\Delta_s}\exp\{(z-H)\lambda_{\beta_1}\}\right)\cos\zeta x\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\zeta, \tag{4·6}
$$

where $$\Delta_s = \Delta_p, \tag{4·7}$$
and

$$\begin{aligned}
4\lambda_{\beta_1}\Delta_{A'}\exp\{-(z-H)\lambda_{\alpha_1}\} \\
= 4\exp\{-z\lambda_{\alpha_1}+H(\lambda_{\alpha_1}+\lambda_{\beta_1})\}\,[&\exp\{-h\lambda_{\beta_1}\}\,2\zeta\lambda_{\beta_1}(2\zeta^2-\kappa^2_{\beta_1})\,S \\
&-\exp\{-h\lambda_{\beta_1}-2H\lambda_{\beta_1}\}\,2\zeta\lambda_{\beta_1}(2\zeta^2-\kappa^2_{\beta_1})\,T \\
&\mp\tfrac{1}{2}\exp\{\pm h\lambda_{\beta_1}-2H\lambda_{\beta_1}\}\,4\zeta\lambda_{\beta_1}(2\zeta^2-\kappa^2_{\beta_1})\,T \\
&+\exp\{-h\lambda_{\beta_1}-H(\lambda_{\alpha_1}+\lambda_{\beta_1})\}\,4\zeta^2\lambda_{\alpha_1}\lambda_{\beta_1}\,V \\
&\pm\tfrac{1}{2}\exp\{\pm h\lambda_{\beta_1}-H(\lambda_{\alpha_1}+\lambda_{\beta_1})\}\,\overline{F}V],
\end{aligned} \tag{4·8}$$

$$\begin{aligned}
4\lambda_{\beta_1}\Delta_{B'}\exp\{(z-H)\lambda_{\alpha_1}\} \\
= 4\exp\{z\lambda_{\alpha_1}+H(\lambda_{\alpha_1}+\lambda_{\beta_1})\}\quad[&\exp\{-h\lambda_{\beta_1}-H(\lambda_{\alpha_1}+\lambda_{\beta_1})\}\,4\zeta^2\lambda_{\alpha_1}\lambda_{\beta_1}\,V \\
&\mp\tfrac{1}{2}\exp\{-h\lambda_{\beta_1}-H(\lambda_{\alpha_1}+\lambda_{\beta_1})\}\,FV \\
&-\exp\{-h\lambda_{\beta_1}-2H\lambda_{\alpha_1}\}\,2\zeta\lambda_{\beta_1}(2\zeta^2-\kappa^2_{\beta_1})\,W \\
&-\exp\{-h\lambda_{\beta_1}-2H(\lambda_{\alpha_1}+\lambda_{\beta_1})\}\,2\zeta\lambda_{\beta_1}(2\zeta^2-\kappa^2_{\beta_1})\,U \\
&\mp\tfrac{1}{2}\exp\{\pm h\lambda_{\beta_1}-2H(\lambda_{\alpha_1}+\lambda_{\beta_1})\}\,4\zeta\lambda_{\beta_1}(2\zeta^2-\kappa^2_{\beta_1})\,U],
\end{aligned} \tag{4·9}$$

$$\begin{aligned}
4\lambda_{\beta_1}\Delta_{C'}\exp\{-(z-H)\lambda_{\beta_1}\} \\
= 4\exp\{-z\lambda_{\beta_1}+H(\lambda_{\alpha_1}+\lambda_{\beta_1})\}\,[&\exp\{-h\lambda_{\beta_1}\}\,4\zeta^2\lambda_{\alpha_1}\lambda_{\beta_1}\,S \\
&\mp\tfrac{1}{2}\exp\{\pm h\lambda_{\beta_1}-2H\lambda_{\beta_1}\}\,\overline{F}T \\
&+\exp\{-h\lambda_{\beta_1}-H(\lambda_{\alpha_1}+\lambda_{\beta_1})\}\,2\zeta\lambda_{\beta_1}(2\zeta^2-\kappa^2_{\beta_1})\,V \\
&\pm\tfrac{1}{2}\exp\{\pm h\lambda_{\beta_1}-H(\lambda_{\alpha_1}+\lambda_{\beta_1})\}\,4\zeta\lambda_{\alpha_1}(2\zeta^2-\kappa^2_{\beta_1})\,Y \\
&+\exp\{-h\lambda_{\beta_1}-2H\lambda_{\alpha_1}\}\,4\zeta^2\lambda_{\alpha_1}\lambda_{\beta_1}W \\
&\pm\tfrac{1}{2}\exp\{\pm h\lambda_{\beta_1}-2H(\lambda_{\alpha_1}+\lambda_{\beta_1})\}\,FU],
\end{aligned} \tag{4·10}$$

$$\begin{aligned}
4\lambda_{\beta_1}\Delta_{D'}\exp\{(z-H)\lambda_{\beta_1}\} \\
= 4\exp\{z\lambda_{\beta_1}+H(\lambda_{\alpha_1}+\lambda_{\beta_1})\}\quad[&-\exp\{-h\lambda_{\beta_1}-2H\lambda_{\beta_1}\}\,4\zeta^2\lambda_{\alpha_1}\lambda_{\beta_1}\,T \\
&\pm\tfrac{1}{2}\exp\{\pm h\lambda_{\beta_1}-2H\lambda_{\beta_1}\}\,FT \\
&\mp\tfrac{1}{2}\exp\{\pm h\lambda_{\beta_1}-H(\lambda_{\alpha_1}+\lambda_{\beta_1})\}\,2\zeta\lambda_{\beta_1}(2\zeta^2-\kappa^2_{\beta_1})\,Y \\
&-\exp\{-h\lambda_{\beta_1}-2H\lambda_{\alpha_1}\}\,4\zeta^2\lambda_{\alpha_1}\lambda_{\beta_1}\,U \\
&\mp\tfrac{1}{2}\exp\{\pm h\lambda_{\beta_1}-2H\lambda_{\alpha_1}\}\,\overline{F}U].
\end{aligned} \tag{4·11}$$

5. Bromwich expansion method applied to the evaluation of the integrals

In their present form the integrals (3·38, 3·39) and (4·3 to 4·6) cannot be evaluated exactly or even approximately by any simple method, but we appeal successively to the 'Bromwich expansion method' and the 'Sommerfeld contour distortion'. The latter, devised for the solution of problems in electro-magnetics (Sommerfeld 1909), was first applied to geophysical problems by Jeffreys (1926*a*), and has since had quite extensive application in this field (Jeffreys 1931; Muskat 1933; Pekeris 1948; Lapwood 1949). Bromwich (1916) showed that, in general, expansion in negative powers of exponentials expresses the motion in a series of pulses.

First we must define the radicals λ_{α_1}, λ_{β_1}, λ_{α_2}, λ_{β_2} more exactly, for when the signs of these are unrestricted the integrands are sixteen-valued functions of ζ. It may be that owing to the special form of the integrand two or more of these values coincide, but in general they require a sixteen-leaved Riemann surface for their representation. We shall confine ourselves to the 'upper leaf' for which $\mathscr{R}(\lambda_{\alpha_1})$, $\mathscr{R}(\lambda_{\beta_1})$, $\mathscr{R}(\lambda_{\alpha_2})$, $\mathscr{R}(\lambda_{\beta_2})$ are all $\geqslant 0$, consistent with (3·8) and the vanishing of the displacements at great depths.

The branch points are the points $\zeta = \pm\kappa_{\alpha_1}$, $\pm\kappa_{\beta_1}$, $\pm\kappa_{\alpha_2}$, $\pm\kappa_{\beta_2}$ at which λ_{α_1}, λ_{β_1}, ... are zero and the lines along which the leaves coalesce are the lines $\mathscr{R}(\lambda_{\alpha_1}) = 0$, etc. For a complex ω which we shall write as $s - ic$ ($s \gtrless 0$, $c > 0$, since Ω or the equivalent contour lies below the real axis), the line $\mathscr{R}(\lambda_{\alpha_1}) = 0$ is given by

$$\lambda_{\alpha_1^2} = \text{(real and negative)},$$

or writing $\zeta = \xi + i\eta$,

$$\left.\begin{aligned} 2\xi\eta + 2sc/\alpha_1^2 &= 0, \\ \xi^2 - \eta^2 &< (s^2 - c^2)/\alpha_1^2. \end{aligned}\right\} \tag{5·1}$$

Equations (5·1) represent arcs of hyperbolas through $\pm\kappa_{\alpha_1}$ and the appropriate arcs, with those corresponding to the λ_{β_1}, λ_{α_2} and λ_{β_2} branch lines, are illustrated in figure 2 for the cases $\mathscr{R}(\omega) \gtrless 0$.

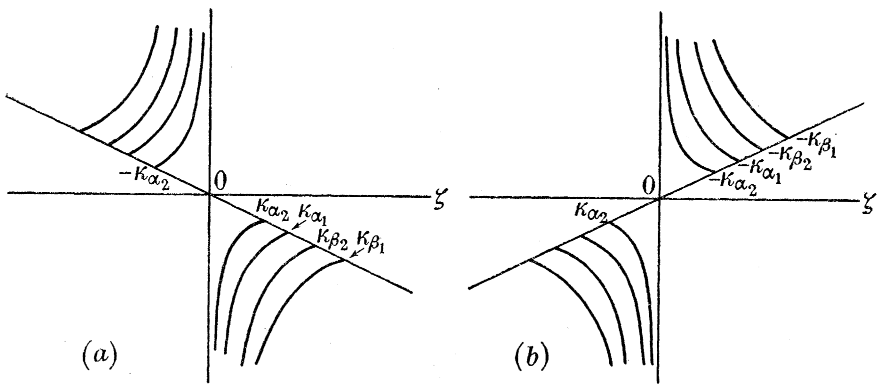

FIGURE 2. Branch lines in ζ-plane. (a) $\mathscr{R}(\omega) > 0$, (b) $\mathscr{R}(\omega) < 0$.

It is convenient at this stage to say something about the density and elastic constants. Just as the disturbance pattern depends only on the ratio of $h:H$ and not on their actual magnitudes so it is governed by the ratios $\rho_2:\rho_1$, $\mu_2:\mu_1$ and $\lambda_2:\lambda_1$. We shall be particularly interested in attempting to relate our results to the records of shallow-focus earthquakes believed to have occurred in the granitic layer overlying a considerable depth of ultra-basic rock. Neglecting the curvature of the earth and any sedimentary deposits near the surface, a two-layer system provides a convenient model for such a comparison and accordingly we take

$$\left.\begin{aligned} \rho_2/\rho_1 &= \tfrac{5}{4}, \\ \mu_2/\mu_1 &= \tfrac{20}{9}, \end{aligned}\right\} \tag{5·2}$$

with Poisson's condition,

$$\lambda = \mu. \tag{5·3}$$

Jeffreys (1935) uses (5·2), and experiments have shown Poisson's condition to be reasonably true for rocks not too far below the surface (see Birch & Law 1935). It will be seen later how little our main results depend on exact values of constants. With the above ratios we have

$$\beta_1 = \alpha_1/\sqrt{3} = \tfrac{3}{4}\beta_2 = \tfrac{3}{4}\alpha_2/\sqrt{3},$$

so that
$$\alpha_2 > \alpha_1 > \beta_2 > \beta_1, \tag{5·4}$$
or
$$|\kappa_{\alpha_2}| < |\kappa_{\alpha_1}| < |\kappa_{\beta_2}| < |\kappa_{\beta_1}|. \tag{5·5}$$

Returning now to the integrals (3·40), (3·41), (4·4) to (4·6) it is seen on closer inspection that ϕ_p, ψ_p are even in λ_{β_1},* so that regarding ζ as a complex variable the arc $\mathscr{R}(\lambda_{\beta_1}) = 0$ is not in fact a branch line of the integrand; this is not so of the arcs $\mathscr{R}(\lambda_{\alpha_1})$, $\mathscr{R}(\lambda_{\alpha_2})$, $\mathscr{R}(\lambda_{\beta_2}) = 0$, and in any distortion of the ζ-path of integration these arcs and any poles of the integrands are to be avoided. The pair ϕ_s, ψ_s are even in λ_{α_1} only, and it is the cuts $\mathscr{R}(\lambda_{\beta_1})$, $\mathscr{R}(\lambda_{\alpha_2})$, $\mathscr{R}(\lambda_{\beta_2}) = 0$ together with any poles which must be avoided. Further, we see that each of the integrals in the ζ-plane, for a given complex ω, is of one of the two forms

$$\chi_1 = \int_0^\infty G(\zeta) \cos \zeta x \, d\zeta, \tag{5·6}$$

$$\chi_2 = \int_0^\infty \zeta G(\zeta) \sin \zeta x \, d\zeta, \tag{5·7}$$

where $G(\zeta)$ is even in ζ and contains a factor vanishing exponentially on any arc of the circle at infinity, except possibly in the neighbourhood of the negative imaginary axis where $G(\zeta)$ is $O(1/|\zeta|)$. By suitable manipulation and distortion of the ζ-contour, Lapwood (1949, p. 74) showed that χ_1 and χ_2 may otherwise be written

$$\mathscr{R}(\omega) > 0: \quad \left. \begin{aligned} \chi_1 &= \frac{1}{2}\int_\Gamma G(\zeta)\, e^{-i\zeta x}\, d\zeta, \\ \chi_2 &= -\frac{1}{2i}\int_\Gamma \zeta G(\zeta)\, e^{-i\zeta x}\, d\zeta, \end{aligned} \right\} \tag{5·8}$$

$$\mathscr{R}(\omega) < 0: \quad \left. \begin{aligned} \chi_1 &= \frac{1}{2}\int_{\Gamma'} G(\zeta)\, e^{-i\zeta x}\, d\zeta, \\ \chi_2 &= \frac{1}{2i}\int_{\Gamma'} \zeta G(\zeta)\, e^{i\zeta x}\, d\zeta, \end{aligned} \right\} \tag{5·9}$$

where Γ and Γ' are large loops in the fourth and first quadrants respectively surrounding all the singular lines and poles of the integrands therein.

For the initial P-pulse, Γ and Γ' must surround the cuts $\mathscr{R}(\lambda_{\alpha_1})$, $\mathscr{R}(\lambda_{\alpha_2})$, $\mathscr{R}(\lambda_{\beta_2}) = 0$ and all the poles of Δ_p of which κ_Δ in figure 3 is supposed to be typical. We saw that this denominator may be written

$$\begin{aligned} \Delta_p = \exp\{H(\lambda_{\alpha_1}+\lambda_{\beta_1})\}\,[S' + T'\exp\{-2H\lambda_{\beta_1}\} + (V'+Y')\exp\{-H(\lambda_{\alpha_1}+\lambda_{\beta_1})\} \\ + W'\exp\{-2H\lambda_{\alpha_1}\} + U'\exp\{-2H(\lambda_{\alpha_1}+\lambda_{\beta_1})\}]. \end{aligned} \tag{5·10}$$

Now suppose that the ω-contour is distorted from the line $-\infty - ic$ to $\infty - ic$ into a large semicircle below the real axis, so that everywhere $\mathscr{I}(\omega) < 0$ and $|\omega|$ is great. Let the radius of this semicircle tend to infinity; then, by our choice of that leaf of the Riemann surface on which $\mathscr{R}(\lambda_{\alpha_1})$, $\mathscr{R}(\lambda_{\beta_1}) \geqslant 0$, we ensure that as close to the cuts $\mathscr{R}(\lambda_{\alpha_1}) = 0$, $\mathscr{R}(\lambda_{\beta_1}) = 0$ as we like but not actually on them

$$\begin{aligned} |T'\exp\{-2H\lambda_{\beta_1}\} + (V'+Y')\exp\{-H(\lambda_{\alpha_1}+\lambda_{\beta_1})\} + W'\exp\{-2H\lambda_{\alpha_1}\} \\ + U'\exp\{-2H(\lambda_{\alpha_1}+\lambda_{\beta_1})\}| \ll |S'|. \end{aligned} \tag{5·11}$$

* To see this recombine the λ_{α_1}, λ_{β_1} exponentials into sinh and cosh terms.

Thus Δ_p may be written

$$\Delta_p = \exp\{H(\lambda_{\alpha_1}+\lambda_{\beta_1})\}S'\Big[1+\frac{T'}{S'}\exp\{-2H\lambda_{\beta_1}\}+\frac{V'+Y'}{S'}\exp\{-H(\lambda_{\alpha_1}+\lambda_{\beta_1})\} + \frac{W'}{S'}\exp\{-2H\lambda_{\alpha_1}\}+\frac{U'}{S'}\exp\{-2H(\lambda_{\alpha_1}+\lambda_{\beta_1})\}\Big], \quad (5\cdot12)$$

and expanded in negative powers of the exponentials, and since any one term is regular on either side of the cuts it may be evaluated there. Also the ω-contour may be redistorted back to Ω or any other equivalent form convenient for the evaluation of a particular term.

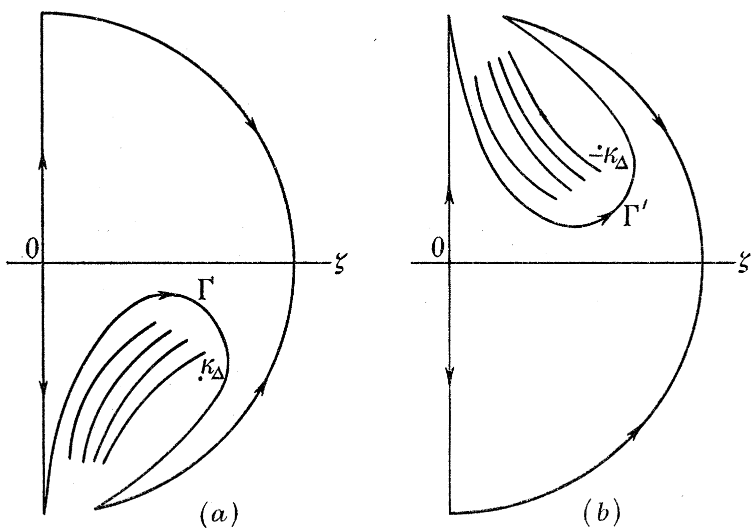

FIGURE 3. Distorted ζ-contour, (a) $\mathscr{R}(\omega)>0$, (b) $\mathscr{R}(\omega)<0$.

Further, because the zeros of Δ_p can only lie inside the loops (i.e. on the cuts) except for the zeros of S', the only relevant poles are the zeros of S', that is, the roots of

$$F(\zeta) = 0, \quad (5\cdot13)$$

and

$$S(\zeta) = 0. \quad (5\cdot14)$$

The equation (5·13) is the phase-velocity equation for Rayleigh waves in medium I $(\rho_1, \mu_1, \lambda_1)$, and it can be shown that it has only one root on the upper leaf of the Riemann surface. We shall denote it by κ_{γ_1}, where $\kappa_{\gamma_1} = \omega/\gamma_1$ and $\gamma_1 = 0{\cdot}9194\ldots\beta_1$.

The equation (5·14) is the 'Stoneley equation' which determines the velocity of waves propagated along the interface between infinite depths of mediums I and II. This will be discussed later, but we let κ_δ denote a possible root of the equation. Expanding the integrands for $\phi_p, \psi_p, \phi_s, \psi_s$ we see that any one term is also of the form (5·6) or (5·7), so that the relations (5·8) and (5·9) still apply where now the loops Γ, Γ' must surround the four cuts $\mathscr{R}(\lambda_{\alpha_1}), \mathscr{R}(\lambda_{\beta_1}), \mathscr{R}(\lambda_{\alpha_2}), \mathscr{R}(\lambda_{\beta_2}) = 0$ and the zeros of $F(\zeta)$ and $S(\zeta)$. It is convenient to replace Γ by four narrow loops $\Gamma_{\alpha_1}, \Gamma_{\beta_1}, \Gamma_{\alpha_2}, \Gamma_{\beta_2}$ close to the cuts as shown and small circles round the poles Γ_{γ_1}, etc. By so doing we shall concentrate in a few regions, namely, the neighbourhood of the branch points and the poles, the main contributions from the integrals.

It is interesting to note that whereas it was remarked earlier that $\mathscr{R}(\lambda_{\beta_1}) = 0$ was not a branch-line of the ϕ_p, ψ_p integrands, we are now avoiding both the λ_{α_1} and λ_{β_1} cuts from considerations of the validity of the series expansions; in agreement with this, we see that although the complete integrands for ϕ_p, ψ_p were even in λ_{β_1}, any single term in the series is not. Thus we may expect definite contributions from integration round the loops Γ_{α_1}, Γ_{β_1}. The same is true of the ϕ_s, ψ_s integrals. The expressions for the potentials may now be written:

(*a*) *For an initial P-pulse* $(z \leqslant h, \mathscr{R}(\omega) \geqslant 0)$:

$$\phi_p = \frac{1}{2}\int_\Gamma 2(\exp\{-(h+z)\,\lambda_{\alpha_1}\} - \exp\{-(h-z)\,\lambda_{\alpha_1}\})\,e^{i\omega t - i\zeta x}\frac{d\zeta}{\lambda_{\alpha_1}}$$

$$+\frac{1}{2}\int_\Gamma e^{i\omega t - i\zeta x}\times \begin{array}{ll} (1) & 16\zeta^2\lambda_{\beta_1}\dfrac{1}{F}\exp\{-(h+z)\,\lambda_{\alpha_1}\} \\ (2) & +32\zeta^2\lambda_{\beta_1}(2\zeta^2-\kappa_{\beta_1}^2)^2\dfrac{1}{F^2}\dfrac{T}{S}\exp\{-(h+z)\,\lambda_{\alpha_1}-2H\lambda_{\beta_1}\} \\ (3) & -\dfrac{8\zeta\lambda_{\beta_1}(2\zeta^2-\kappa_{\beta_1}^2)}{\lambda_{\alpha_1}}\dfrac{1}{F}\dfrac{Y}{S}\exp\{-(H+h-z)\,\lambda_{\alpha_1}-H\lambda_{\beta_1}\} \\ (4) & -\dfrac{8\zeta\lambda_{\beta_1}(2\zeta^2-\kappa_{\beta_1}^2)}{\lambda_{\alpha_1}}\dfrac{1}{F}\dfrac{Y}{S}\exp\{-(H-h+z)\,\lambda_{\alpha_1}-H\lambda_{\beta_1}\} \\ (5) & +\dfrac{16\zeta\lambda_{\beta_1}(2\zeta^2-\kappa_{\beta_1}^2)}{\lambda_{\alpha_1}}\dfrac{\bar{F}}{F^2}\dfrac{Y}{S}\exp\{-(H+h+z)\,\lambda_{\alpha_1}-H\lambda_{\beta_1}\} \\ (6) & +\dfrac{2}{\lambda_{\alpha_1}}\dfrac{W}{S}\exp\{-(2H-h-z)\,\lambda_{\alpha_1}\} \\ (7) & -\dfrac{2}{\lambda_{\alpha_1}}\dfrac{\bar{F}}{F}\dfrac{W}{S}\exp\{-(2H+h-z)\,\lambda_{\alpha_1}\} \\ (8) & -\dfrac{2}{\lambda_{\alpha_1}}\dfrac{\bar{F}}{F}\dfrac{W}{S}\exp\{-(2H-h+z)\,\lambda_{\alpha_1}\} \\ (9) & +\dfrac{2}{\lambda_{\alpha_1}}\dfrac{\bar{F}^2}{F^2}\dfrac{W}{S}\exp\{-(2H+h+z)\,\lambda_{\alpha_1}\}\,d\zeta \end{array}$$

$$+\text{terms containing } 4H,\ 6H,\ \ldots \text{ in the exponents;} \tag{5·15}$$

$$\psi_p = -\frac{1}{2i}\int_\Gamma e^{i\omega t - i\zeta x}\times \begin{array}{ll} (1) & -8\zeta(2\zeta^2-\kappa_{\beta_1}^2)\dfrac{1}{F}\exp\{-h\lambda_{\alpha_1}-z\lambda_{\beta_1}\} \\ (2) & +8\zeta(2\zeta^2-\kappa_{\beta_1}^2)\dfrac{1}{F}\dfrac{T}{S}\exp\{-h\lambda_{\alpha_1}-(2H-z)\,\lambda_{\beta_1}\} \\ (3) & -8\zeta(2\zeta^2-\kappa_{\beta_1}^2)\dfrac{\bar{F}}{F^2}\dfrac{T}{S}\exp\{-h\lambda_{\alpha_1}-(2H+z)\,\lambda_{\beta_1}\} \\ (4) & -\dfrac{2}{\lambda_{\alpha_1}}\dfrac{Y}{S}\exp\{-(H-h)\,\lambda_{\alpha_1}-(H-z)\,\lambda_{\beta_1}\} \\ (5) & +\dfrac{2}{\lambda_{\alpha_1}}\dfrac{\bar{F}}{F}\dfrac{Y}{S}\exp\{-(H-h)\,\lambda_{\alpha_1}-(H+z)\,\lambda_{\beta_1}\} \\ (6) & +\dfrac{2}{\lambda_{\alpha_1}}\dfrac{\bar{F}}{F}\dfrac{Y}{S}\exp\{-(H+h)\,\lambda_{\alpha_1}-(H-z)\,\lambda_{\beta_1}\} \\ (7) & -\dfrac{2}{\lambda_{\alpha_1}}\dfrac{\bar{F}^2}{F^2}\dfrac{Y}{S}\exp\{-(H+h)\,\lambda_{\alpha_1}-(H+z)\,\lambda_{\beta_1}\} \\ (8) & +8\zeta(2\zeta^2-\kappa_{\beta_1}^2)\dfrac{1}{F}\dfrac{W}{S}\exp\{-(2H-h)\,\lambda_{\alpha_1}-z\lambda_{\beta_1}\} \\ (9) & -8\zeta(2\zeta^2-\kappa_{\beta_1}^2)\dfrac{\bar{F}}{F^2}\dfrac{W}{S}\exp\{-(2H+h)\,\lambda_{\alpha_1}-z\lambda_{\beta_1}\}\,d\zeta \end{array}$$

$$+\text{terms containing } 4H,\ 6H,\ \ldots \text{ in the exponents.} \tag{5·16}$$

(*b*) *For initial S-pulse* ($z \leqslant h$, $\mathscr{R}(\omega) \geqslant 0$):

$$\phi_s = -\frac{1}{2\mathrm{i}}\int_\Gamma \mathrm{e}^{\mathrm{i}\omega t - \mathrm{i}\zeta x} \times \begin{aligned}[t] &(1)\quad 8\zeta(2\zeta^2-\kappa_{\beta_1}^2)\frac{1}{F}\exp\{-z\lambda_{\alpha_1}-h\lambda_{\beta_1}\}\\ &(2)\quad -8\zeta(2\zeta^2-\kappa_{\beta_1}^2)\frac{1}{F}\frac{T}{S}\exp\{-z\lambda_{\alpha_1}-(2H-h)\,\lambda_{\beta_1}\}\\ &(3)\quad +8\zeta(2\zeta^2-\kappa_{\beta_1}^2)\frac{\overline{F}}{F^2}\frac{T}{S}\exp\{-z\lambda_{\alpha_1}-(2H+h)\,\lambda_{\beta_1}\}\\ &(4)\quad +\frac{2}{\lambda_{\alpha_1}}\frac{Y}{S}\exp\{-(H-z)\,\lambda_{\alpha_1}-(H-h)\,\lambda_{\beta_1}\}\\ &(5)\quad -\frac{2}{\lambda_{\alpha_1}}\frac{\overline{F}}{F}\frac{Y}{S}\exp\{-(H-z)\,\lambda_{\alpha_1}-(H+h)\,\lambda_{\beta_1}\}\\ &(6)\quad -\frac{2}{\lambda_{\alpha_1}}\frac{\overline{F}}{F}\frac{Y}{S}\exp\{-(H+z)\,\lambda_{\alpha_1}-(H-h)\,\lambda_{\beta_1}\}\\ &(7)\quad +\frac{2}{\lambda_{\alpha_1}}\frac{\overline{F}^2}{F^2}\frac{Y}{S}\exp\{-(H+z)\,\lambda_{\alpha_1}-(H+h)\,\lambda_{\beta_1}\}\\ &(8)\quad -8\zeta(2\zeta^2-\kappa_{\beta_1}^2)\frac{1}{F}\frac{W}{S}\exp\{-h\lambda_{\beta_1}-(2H-z)\lambda_{\alpha_1}\}\\ &(9)\quad +8\zeta(2\zeta^2-\kappa_{\beta_1}^2)\frac{\overline{F}}{F^2}\frac{W}{S}\exp\{-h\lambda_{\beta_1}-(2H+z)\,\lambda_{\alpha_1}\}\,\mathrm{d}\zeta \end{aligned}$$

$$+\dots; \tag{5·17}$$

$$\psi_s = \frac{1}{2}\int_\Gamma 2(\exp\{-(h+z)\,\lambda_{\beta_1}\}-\exp\{-(h-z)\,\lambda_{\beta_1}\})\,\mathrm{e}^{\mathrm{i}\omega t-\mathrm{i}\zeta x}\frac{\mathrm{d}\zeta}{\lambda_{\beta_1}}$$

$$+\frac{1}{2}\int_\Gamma \mathrm{e}^{\mathrm{i}\omega t - \mathrm{i}\zeta x} \times \begin{aligned}[t] &(1)\quad 16\zeta^2\lambda_{\alpha_1}\frac{1}{F}\exp\{-(h+z)\,\lambda_{\beta_1}\}\\ &(2)\quad +32\zeta^2\lambda_{\alpha_1}(2\zeta^2-\kappa_{\beta_1}^2)^2\frac{1}{F^2}\frac{T}{S}\exp\{-(h+z)\,\lambda_{\beta_1}-2H\lambda_{\alpha_1}\}\\ &(3)\quad -8\zeta(2\zeta^2-\kappa_{\beta_1}^2)\frac{1}{F}\frac{Y}{S}\exp\{-H\lambda_{\alpha_1}-(H+h-z)\,\lambda_{\beta_1}\}\\ &(4)\quad -8\zeta(2\zeta^2-\kappa_{\beta_1}^2)\frac{1}{F}\frac{Y}{S}\exp\{-H\lambda_{\alpha_1}-(H-h+z)\,\lambda_{\beta_1}\}\\ &(5)\quad +16\zeta(2\zeta^2-\kappa_{\beta_1}^2)\frac{\overline{F}}{F^2}\frac{Y}{S}\exp\{-H\lambda_{\alpha_1}-(H+h+z)\,\lambda_{\beta_1}\}\\ &(6)\quad \frac{2}{\lambda_{\beta_1}}\frac{T}{S}\exp\{-(2H-h-z)\,\lambda_{\beta_1}\}\\ &(7)\quad -\frac{2}{\lambda_{\beta_1}}\frac{\overline{F}}{F}\frac{T}{S}\exp\{-(2H+h-z)\,\lambda_{\beta_1}\}\\ &(8)\quad -\frac{2}{\lambda_{\beta_1}}\frac{\overline{F}}{F}\frac{T}{S}\exp\{-(2H-h+z)\,\lambda_{\beta_1}\}\\ &(9)\quad +\frac{2}{\lambda_{\beta_1}}\frac{\overline{F}^2}{F^2}\frac{T}{S}\exp\{-(2H+h+z)\,\lambda_{\beta_1}\}\,\mathrm{d}\zeta \end{aligned}$$

$$+\dots, \tag{5·18}$$

Those terms independent of H (and incidentally of the properties of the lower medium) we shall call zero-order terms, and those with $2nH$ in the exponent we shall call nth-order terms.

We notice the similarity of the expressions for ϕ_p and ψ_s, and ψ_p and ϕ_s respectively, the one related to the other by interchange of λ_{α_1} and λ_{β_1}, h and z and substitution of $-\mathrm{i}$ for $+\mathrm{i}$, but we shall see that it is just such an interchange of λ_{α_1}, λ_{β_1} which can alter the whole form of the associated disturbance.

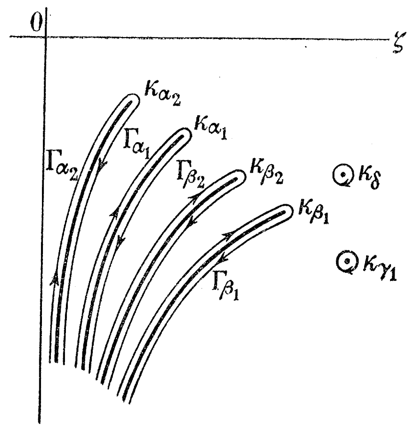

FIGURE 4. 'Sommerfeld contour' in the ζ-plane, $\mathscr{R}(\omega) > 0$.

It is understood in the above that Γ is the composite contour shown in figure 4 and that Φ_p, Ψ_p, Φ_s, Ψ_s are derived by subsequent integration with respect to ω along Ω or any equivalent contour. When $\mathscr{R}(\omega) < 0$ the integrands are the same but for a change of sign in ψ_p and ϕ_s, the substitution of $e^{i\zeta x}$ for $e^{-i\zeta x}$ and the change of contour from Γ to Γ'.

Thus, $\mathscr{R}(\omega) < 0$:

$$\begin{cases} \phi_p = \dfrac{1}{2}\displaystyle\int_{\Gamma'} e^{i\omega t + i\zeta x} \times \ldots \, d\zeta, & (5\cdot19) \\ \psi_p = \dfrac{1}{2i}\displaystyle\int_{\Gamma'} e^{i\omega t + i\zeta x} \times \ldots \, d\zeta, & (5\cdot20) \end{cases}$$

$$\begin{cases} \phi_s = \dfrac{1}{2i}\displaystyle\int_{\Gamma'} e^{i\omega t + i\zeta x} \times \ldots \, d\zeta, & (5\cdot21) \\ \psi_s = \dfrac{1}{2}\displaystyle\int_{\Gamma'} e^{i\omega t + i\zeta x} \times \ldots \, d\zeta. & (5\cdot22) \end{cases}$$

We shall attempt to interpret, as pulses, the terms quoted, and later either extend our interpretation to the 'higher-order' terms or else justify their neglect. The contributions from each of the loops Γ_{α_2}, Γ_{α_1}, Γ_{β_2}, Γ_{β_1} and the circle Γ_{γ_1} will be evaluated in turn, but we shall precede the actual determinations by one or two general considerations.

6. THE ζ-INTEGRATIONS ON Γ_{α_2}, Γ_{α_1}, Γ_{β_2}, Γ_{β_1}: GENERAL CONSIDERATIONS

For the purpose of the following discussion we shall suppose $\mathscr{R}(\omega) > 0$, and that we require the contribution from the loop Γ_{α_1}. A general term of the expressions for ϕ_p, etc., may be written

$$\frac{1}{2}\int_{\Gamma_{\alpha_1}} G(\zeta) \exp\{-i\zeta x - h_1\lambda_{\alpha_1} - h_2\lambda_{\beta_1}\} \, d\zeta, \tag{6·1}$$

where h_1 and h_2 are linear forms in h, z, H, and $G(\zeta)$ is either odd or even in ζ. Consider a general complex ω such that $\mathscr{R}(\omega)>0$ and $|\omega|$ is sufficiently large for the exponential factor to vary much faster than the remainder of the integrand; also x sufficiently great compared with h_1 and h_2 for the part $e^{-ix\zeta}$ to dominate $\exp\{-h_1\lambda_{\alpha_1}-h_2\lambda_{\beta_1}\}$ in the neighbourhood of the branch point κ_{α_1}. Then since $\mathscr{I}(\zeta)$ becomes increasingly negative as we recede from the branch point along either side of the loop Γ_{α_1}, the main contribution to the integrand must come from the neighbourhood of κ_{α_1} and so must be governed by a factor $\exp\{i\omega t-i\kappa_{\alpha_1}x\}$. It would appear then to represent a pulse which has travelled most of the way from the source to the observer with velocity α_1. Similar interpretations may be put on the Γ_{β_1}, Γ_{α_2}, Γ_{β_2} integrals. Further, we may deduce the initial and final type of a pulse, but this qualitative description becomes increasingly incomplete for higher-order terms.

In order to evaluate the expressions like (6·1) we shall introduce a new variable on the cuts. For example, on Γ_{α_1} we shall put

$$\lambda_{\alpha_1}=\pm iu, \tag{6·2}$$

where, for $\mathscr{R}(\omega)>0$, since both $\mathscr{R}(\kappa_{\alpha_1})$, $\mathscr{R}(\lambda_{\alpha_1})$ must be $\geqslant 0$, we see that $\mathscr{I}(\lambda_{\alpha_1})$ must be positive on the left-hand side of the cut and negative on the right; vice versa for $\mathscr{R}(\omega)<0$. The main contribution comes from the neighbourhood of $u=0$. Neglecting higher order terms in u we may write a determinant T, say, as

$$T_{(\text{on }\Gamma_{\alpha_1})} \doteqdot T_{0,\alpha_1}\pm iuT_{1,\alpha_1} \quad \text{(left-hand and right-hand side of cut)},$$

where T_{0,α_1}, T_{1,α_1} are independent of u, and we may approximate to other coefficients and to the exponents likewise. Then on manipulation the various integrals reduce to one or the sum of types

$$G_1(\kappa_{\alpha_1})\int_0^\infty \exp\left\{\frac{ixu^2}{2\kappa_{\alpha_1}}\right\}\cos(hu)\,du, \tag{6·3}$$

$$G_2(\kappa_{\alpha_1})\int_0^\infty \exp\left\{\frac{ixu^2}{2\kappa_{\alpha_1}}\right\}u\sin(hu)\,du, \tag{6·4}$$

$$G_3(\kappa_{\alpha_1})\int_0^\infty \exp\left\{\frac{ixu^2}{2\kappa_{\alpha_1}}\right\}u^2\cos(hu)\,du, \tag{6·5}$$

which may be integrated exactly. Assuming that x is large, the importance of a contribution is determined by the power of x occurring. An integral of type (6·3) gives a power $x^{-\frac{1}{2}}$. One of types (6·4) or (6·5) gives $x^{-\frac{3}{2}}$, so that (6·5) must not be neglected in comparison with (6·4).

A relation like (6·2) will be used to approximate to each determinant on each of the cuts thus defining

$$S_{0,\beta_1}, \quad S_{1,\beta_1}, \quad W_{0,\alpha_2}, \quad \text{etc.}$$

7. The ω-integrations: general considerations

If $f(\omega)$ represents any term of ϕ_p, ϕ_s, ψ_p or ψ_s as determined above for an initial harmonic vibration of period $2\pi/\omega$, then for an initial unit pulse of the same type we have a contribution

$$\frac{1}{2\pi i}\int_\Omega f(\omega)\frac{d\omega}{\omega}, \tag{7·1}$$

so long as $f(\omega)$ is analytic in the region containing Ω and the integral converges. It will be found that $f(\omega)$ always takes one of the forms

$$(1)\quad \omega^n\,\mathrm{e}^{\mathrm{i}\omega\tau}, \tag{7·2}$$

$$(2)\quad \pm\omega^n\,\mathrm{e}^{\mathrm{i}\omega\tau}, \tag{7·3}$$

$$(3)\quad \omega^n\,\mathrm{e}^{\mathrm{i}\omega\tau\mp p\omega}, \tag{7·4}$$

$$(4)\quad \pm\omega^n\,\mathrm{e}^{\mathrm{i}\omega\tau\mp p\omega}, \tag{7·5}$$

the last three approximations being valid except within a narrow sector $\arg\omega = -\frac{1}{2}\pi\pm\epsilon$, in which $f(\omega)$ makes a continuous transition from the general form for $\mathscr{R}(\omega)>0$ to that for $\mathscr{R}(\omega)<0$.* We cannot avoid this sector if our contour is Ω. If, however, $n>0$ we may replace Ω by Ω' proceeding from $-\infty$ to $+\infty$ through the origin by two loops below the real axis, thus avoiding $\arg\omega$ between $-\frac{1}{2}\pi\pm\epsilon$. Then $f(\omega)$ as given by (7·3) and (7·4) is continuous right up to and through the origin and the integral is seen to converge. If $n \not> 0$ then we shall consider $\dot{\phi}$ or $\dot{u}$ (and derive $\dot{\Phi}$ or $\dot{U}$), for differentiation will raise the power of ω to a positive value and ensure that $f(\omega)$ is analytic and the integral convergent.

Results which we shall use are

$$\frac{1}{2\pi\mathrm{i}}\int_\Omega \omega^{-\frac{1}{2}}\,\mathrm{e}^{\mathrm{i}\omega\tau}\frac{\mathrm{d}\omega}{\omega} = 2(\mathrm{i}\tau)^{\frac{1}{2}}\,\pi^{-\frac{1}{2}}\,H(\tau), \tag{7·6}$$

$$\frac{1}{2\pi\mathrm{i}}\int_\Omega \omega^{-\frac{3}{2}}\,\mathrm{e}^{\mathrm{i}\omega\tau}\frac{\mathrm{d}\omega}{\omega} = \tfrac{4}{3}(\mathrm{i}\tau)^{\frac{3}{2}}\,\pi^{-\frac{1}{2}}\,H(\tau) \tag{7·7}$$

(Copson 1935, p. 226), applicable when $f(\omega)$ is of type (7·2),

$$\frac{1}{2\pi\mathrm{i}}\int_{\Omega'} \pm\omega^{\frac{1}{2}}\,\mathrm{e}^{\mathrm{i}\omega\tau}\frac{\mathrm{d}\omega}{\omega} = -\mathrm{i}^{\frac{1}{2}}(\pi\tau')^{-\frac{1}{2}}\,H(\tau'), \tag{7·8}$$

where $\tau' = -\tau$, applicable when $f(\omega)$ is of type (7·3), and

$$\frac{1}{2\pi\mathrm{i}}\int_{\Omega'} \omega\,\mathrm{e}^{\mp p\omega+\mathrm{i}\omega\tau}\frac{\mathrm{d}\omega}{\omega} = \frac{1}{\pi\mathrm{i}}\frac{p}{p^2+\tau^2}, \tag{7·9}$$

$$\frac{1}{2\pi\mathrm{i}}\int_{\Omega'} \pm\omega\,\mathrm{e}^{\mp p\omega+\mathrm{i}\omega\tau}\frac{\mathrm{d}\omega}{\omega} = \frac{1}{\pi}\frac{\tau}{p^2+\tau^2}, \tag{7·10}$$

$$\frac{1}{2\pi\mathrm{i}}\int_{\Omega'} \omega^{\frac{1}{2}}\,\mathrm{e}^{\mp p\omega+\mathrm{i}\omega\tau}\frac{\mathrm{d}\omega}{\omega} = \mathrm{i}^{-\frac{1}{2}}(\pi p)^{-\frac{1}{2}}\cos^{\frac{1}{2}}\psi\,\sin\left(\tfrac{1}{2}\psi+\tfrac{1}{4}\pi\right), \tag{7·11}$$

$$\frac{1}{2\pi\mathrm{i}}\int_{\Omega'} \pm\omega^{\frac{1}{2}}\,\mathrm{e}^{\mp p\omega+\mathrm{i}\omega\tau}\frac{\mathrm{d}\omega}{\omega} = \mathrm{i}^{\frac{1}{2}}(\pi p)^{-\frac{1}{2}}\cos^{\frac{1}{2}}\psi\,\sin\left(\tfrac{1}{2}\psi-\tfrac{1}{4}\pi\right), \tag{7·12}$$

where $\tan\psi = \tau/p$, applicable when $f(\omega)$ is of types (7·4) and (7·5). This second group of relations may be derived from a result of Stewart (1940, p. 503),

$$\int_0^\infty \omega^{n-1}\,\mathrm{e}^{-p\omega}\,{\cos \atop \sin}\,\omega\tau\,\mathrm{d}\omega = \Gamma(n)\,(p^2+\tau^2)^{-\frac{1}{2}n}\,{\cos \atop \sin}\,n\psi, \tag{7·13}$$

where $\psi = \tan^{-1}\tau/p$; they are quoted with modifications from Lapwood (1949, p. 66).

* The explanation of the failure of the approximation lies in the fact that as $\arg\omega\to\frac{1}{2}\pi$ the four loops Γ_{α_2}, Γ_{α_1}, Γ_{β_2}, Γ_{β_1} close up into a single loop from $-\mathrm{i}\infty$ to $-\mathrm{i}\infty$ surrounding κ_{α_2}, κ_{α_1}, κ_{β_2}, κ_{β_1} on the imaginary axis (see appendix 1); contributions from, say, the neighbourhood of κ_{β_1} on the loop Γ_{β_1} then become increasingly influenced by the proximity of Γ_{β_2}, Γ_{α_1} and Γ_{α_2}.

Owing to the nature of the determinants S, T, V, etc., it may be that part of an expression retains the same sign for $\mathscr{R}(\omega) \gtrless 0$ and the remainder changes sign. The two parts must be integrated distinctly according to the results above. That part of a coefficient which retains the same sign will be denoted by a suffix $_0$ and that which changes by suffix $_1$. Thus we shall write

$$T_{0,\beta_1} = (T_0)_{\beta_1,0} \pm (T_0)_{\beta_1,1} \quad (\mathscr{R}(\omega) \gtrless 0), \tag{7·14}$$

$$\left(\frac{T_{1,\alpha_1}}{S_{0,\alpha_1}}\right) = \left(\frac{T_1}{S_0}\right)_{\alpha_1} = \left(\frac{T_1}{S_0}\right)_{\alpha_1,0} + \left(\frac{T_1}{S_0}\right)_{\alpha_1,1} \quad (\mathscr{R}(\omega) \gtrless 0). \tag{7·15}$$

We shall introduce the real positive quantities $\widehat{\beta_1\beta_2}$, $\widehat{\alpha_1\alpha_2}$, ... and $\kappa_{\widehat{\beta_1\beta_2}}$, $\kappa_{\widehat{\alpha_1\alpha_2}}$, ..., where

$$\widehat{\beta_1\beta_2} = (1/\beta_1^2 - 1/\beta_2^2)^{-\frac{1}{2}}, \quad \text{etc.}, \tag{7·16}$$

and

$$\kappa_{\widehat{\beta_1\beta_2}} = \omega/\widehat{\beta_1\beta_2} = (\kappa_{\beta_1}^2 - \kappa_{\beta_2}^2)^{\frac{1}{2}}, \quad \text{etc.} \tag{7·17}$$

It is seen that

(λ_{β_2}) evaluated at the branch-point κ_{β_1} on $\Gamma_{\beta_1} = +\kappa_{\widehat{\beta_1\beta_2}}$ $(\mathscr{R}(\omega) > 0)$

and (λ_{β_2}) evaluated at the branch-point $-\kappa_{\beta_1}$ on $\Gamma'_{\beta_1} = -\kappa_{\beta_1\beta_2}$ $(\mathscr{R}(\omega) < 0)$.

On the other hand,

(λ_{β_1}) evaluated at κ_{β_2} on $\Gamma_{\beta_2} = \mathrm{i}\kappa_{\widehat{\beta_1\beta_2}}$

$= (\lambda_{\beta_1})$ evaluated at $-\kappa_{\beta_2}$ on Γ'_{β_2}.

Thus comparing our various expressions on

(*a*) Γ_{α_2} and Γ'_{α_2}, there is a sign change associated with ζ,

(*b*) Γ_{α_1} and Γ'_{α_1}, there is a sign change associated with ζ and λ_{α_2},

(*c*) Γ_{β_2} and Γ'_{β_2}, there is a sign change associated with ζ, λ_{α_2} and λ_{α_1},

(*d*) Γ_{β_1} and Γ'_{β_1}, there is a sign change associated with ζ, λ_{α_2}, λ_{α_1} and λ_{β_2},

(*e*) Γ_{γ_1} and Γ'_{γ_1}, there is a sign change associated with ζ, λ_{α_2}, λ_{α_1}, λ_{β_2} and λ_{β_1}.

(Until we locate κ_δ (Stoneley pole or poles) exactly, we cannot add a similar statement about Γ_δ and Γ'_δ.)

8. The shape of the initial pulse

By choosing the source to vary in time as a unit Heaviside function we are making a direct extension of Lapwood's work (1949) to a layered medium. Our Φ_0, Ψ_0 (and Φ_r and Ψ_r) are necessarily identical and the reader is referred to an interesting comparison Lapwood made (1949, p. 82) between Φ_0 given exactly and Φ_0 calculated approximately as an integral on Γ_{α_1} (by methods which will be amply illustrated in succeeding sections).

Exactly, we have
$$\Phi_0 = -2\cosh^{-1}\left(\frac{\alpha_1 t}{\varpi}\right) H\left(t - \frac{\varpi}{\alpha_1}\right), \tag{8·1}$$

with the derived radial displacement

$$U_{0,\varpi} = \frac{\partial \Phi_0}{\partial \varpi} = \frac{2\alpha_1 t}{\varpi\sqrt{(\alpha_1^2 t^2 - \varpi^2)}} H\left(t - \frac{\varpi}{\alpha_1}\right), \tag{8·2}$$

or for small $\alpha_1 t/\varpi$,
$$U_{0,\varpi} \doteqdot \sqrt{\left(\frac{2}{\varpi\alpha_1\tau_0}\right)}\left(1+\frac{\alpha_1\tau_0}{\varpi}\right)\left(1+\frac{\alpha_1\tau_0}{2\varpi}\right)^{-\frac{1}{2}} H(\tau_0), \tag{8·3}$$

where
$$\tau_0 = t-\varpi/\alpha_1. \tag{8·4}$$

Figure 5, reproduced from Lapwood (1949, p. 68), shows Φ_0, $U_{0,\varpi}$ as functions of time and radial distance from the source. The infinities at $\varpi = 0$ and $t = \varpi/\alpha_1$ mark the failure of the Hankel function to correspond to physical conditions.

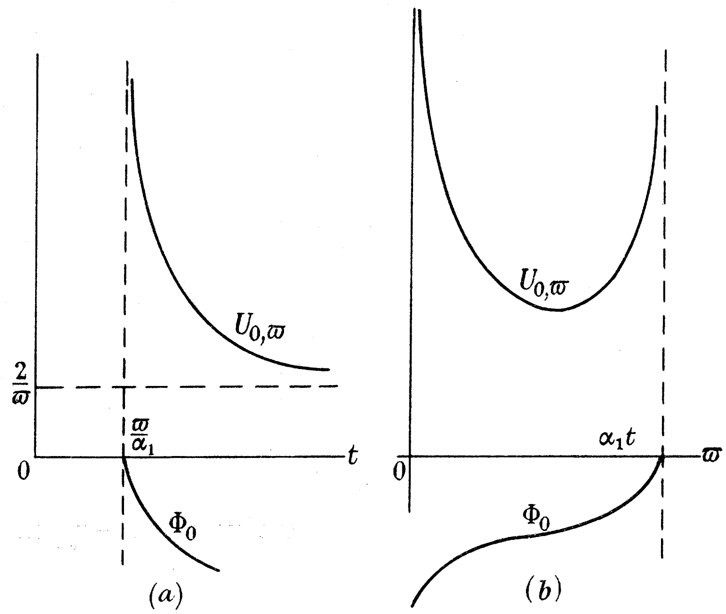

FIGURE 5. Φ_0 and $U_{0,\varpi}$ graphed (*a*) against t, (*b*) against ϖ.

By the approximate method we find
$$\Phi_0 \doteqdot -2\sqrt{\left(\frac{2\alpha_1\tau}{x}\right)}H(\tau) \tag{8·5}$$

with derived displacements (horizontal and vertical)
$$U_0 \doteqdot \sqrt{\left(\frac{2}{x\alpha_1\tau}\right)}H(\tau), \tag{8·6}$$
$$W_0 \doteqdot -\frac{h-z}{x}\sqrt{\left(\frac{2}{x\alpha_1\tau}\right)}H(\tau), \tag{8·7}$$

where $\tau = t-x/\alpha_1-(h-z)^2/2x\alpha_1$.

Comparison of (8·1) to (8·3) and (8·5) to (8·7) shows that we have obtained a good approximation so long as $(h-z)/x$ and $\alpha_1\tau_0/2\varpi$ are small; the essential discrepancy is the failure of (8·6) and (8·7) to record any residual displacement.

We now have some standard by which to assess the accuracy of future approximations.

9. THE EVALUATION OF THE COMPONENTS OF Φ_p, Ψ_p, Φ_s, Ψ_s

From our resolution of ϕ_p, ψ_p, ϕ_s, ψ_s into a succession of pulses it follows naturally to adopt a notation
$$\phi_p = \phi_0+\phi_r+{}_p\phi^{(1)}+{}_p\phi^{(2)}+\dots \tag{9·1}$$
$$\psi_p = {}_p\psi^{(1)}+{}_p\psi^{(2)}+\dots, \quad \text{etc.,} \tag{9·2}$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

where the (1), (2), ... here correspond to the (1), (2), ... in the presentation (5·15) to (5·18). From the total of forty 'zero-order' and 'first-order' terms quoted there, ${}_p\phi^{(4)}$ will be selected and its treatment presented in full. This should serve to illustrate all the essential points arising.

We have

$$\begin{aligned}{}_p\phi^{(4)} &= \frac{1}{2}\int_\Gamma \frac{(-8)\,\zeta\lambda_{\beta_1}(2\zeta^2-\kappa^2_{\beta_1})}{\lambda_{\alpha_1}F}\frac{Y}{S}\exp\{\mathrm{i}\omega t-\mathrm{i}\zeta x-(H-h+z)\,\lambda_{\alpha_1}-H\lambda_{\beta_1}\}\,\mathrm{d}\zeta \quad (\mathscr{R}(\omega)>0),\\ &= \frac{1}{2}\int_{\Gamma'} \frac{(-8)\,\zeta\lambda_{\beta_1}(2\zeta^2-\kappa^2_{\beta_1})}{\lambda_{\alpha_1}F}\frac{Y}{S}\exp\{\mathrm{i}\omega t+\mathrm{i}\zeta x-(H-h+z)\,\lambda_{\alpha_1}-H\lambda_{\beta_1}\}\,\mathrm{d}\zeta \quad (\mathscr{R}(\omega)<0),\end{aligned} \tag{9·3}$$

and

$${}_p\Phi^{(4)} = \frac{1}{2\pi\mathrm{i}}\int_\Omega {}_p\phi^{(4)}\frac{\mathrm{d}\omega}{\omega}. \tag{9·4}$$

Now we shall consider in turn the contributions from the four loops and the circle round the Rayleigh pole $(\pm)\,\kappa_{\gamma_1}$, ignoring for the present the possibility of a root of $S(\zeta)$ and additional pole(s) κ_δ.

(a) *Contribution from* Γ_{α_2}: ${}_p\phi^{(4)}_{\alpha_2}$

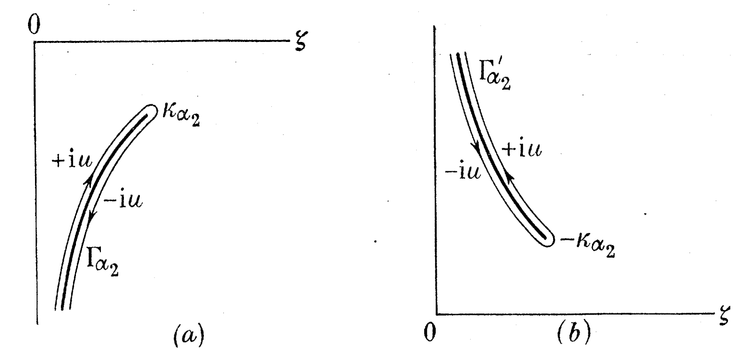

FIGURE 6. Evaluation on Γ_{α_2} and Γ'_{α_2}, (*a*) $\mathscr{R}(\omega)>0$, (*b*) $\mathscr{R}(\omega)<0$.

(i) $\mathscr{R}(\omega)>0$. Since Γ_{α_2} is supposed to lie indefinitely close to the cut $\mathscr{R}(\lambda_{\alpha_2})=0$ (figure 6), we may write on Γ_{α_2},

$\lambda_{\alpha_2}=\pm\mathrm{i}u$ (on the left-hand and right-hand side of the cut respectively—see (6·2)).

Again, if $|x\kappa_{\alpha_2}|$ is sufficiently large, the main contribution to the integral is provided by the neighbourhood of $u=0$.

Thus
$$\zeta=(\lambda^2_{\alpha_2}+\kappa^2_{\alpha_2})^{\frac{1}{2}} \doteqdot \kappa_{\alpha_2}-u^2/2\kappa_{\alpha_2},$$
and
$$\zeta\,\mathrm{d}\zeta=-u\,\mathrm{d}u.$$
Also
$$\begin{aligned}\lambda_{\alpha_1}=(\zeta^2-\kappa^2_{\alpha_1})^{\frac{1}{2}} &\doteqdot \mathrm{i}(\kappa_{\widehat{\alpha_1\alpha_2}}+u^2/2\kappa_{\widehat{\alpha_1\alpha_2}})\\ &\doteqdot \mathrm{i}\kappa_{\widehat{\alpha_1\alpha_2}},\end{aligned}$$
and
$$\lambda_{\beta_1}=(\zeta^2-\kappa^2_{\beta_1})^{\frac{1}{2}} \doteqdot \mathrm{i}\kappa_{\widehat{\beta_1\alpha_2}}.$$
Write
$$Y \doteqdot Y_{0,\,\alpha_2}\pm\mathrm{i}uY_{1,\,\alpha_2},\quad \text{etc.}$$

Taking second approximations to λ_{α_1}, λ_{β_1} in the exponents but only first approximations elsewhere we find

$$
\begin{aligned}
{}_p\phi^{(4)}_{\alpha_2} \doteqdot & -\frac{1}{2}\int_0^\infty (-8\mathrm{i})\,\frac{\kappa_{\widehat{\beta_1\alpha_2}}(2\kappa^2_{\alpha_2}-\kappa^2_{\beta_1})}{\mathrm{i}\kappa_{\widehat{\alpha_1\alpha_2}}F_{0,\,\alpha_2}}\;\mathrm{i}u\left(\frac{Y_1}{S_0}-\frac{Y_0S_1}{S_0^2}\right)_{\alpha_2} \\
& \times\exp\{\mathrm{i}\omega(t-x/\alpha_2-(H-h+z)/\widehat{\alpha_1\alpha_2}-H/\widehat{\beta_1\alpha_2}) \\
& \quad -\mathrm{i}u^2(x/2\kappa_{\alpha_2}-(H-h+z)/2\kappa_{\widehat{\alpha_1\alpha_2}}-H/2\kappa_{\widehat{\beta_1\alpha_2}})\}\,(-u)\,\mathrm{d}u \\
& +\frac{1}{2}\int_0^\infty \text{(same integrand but with } (-\mathrm{i}u) \text{ in place of } (+\mathrm{i}u))\,\mathrm{d}u,
\end{aligned} \tag{9·5}
$$

since $F_{1,\,\alpha_2}\equiv 0$ and the lowest-order term containing $(Y_0/S_0)_{\alpha_2}$ takes the same value on either side of the cut and gives no contribution.

Thus

$$
{}_p\phi^{(4)}_{\alpha_2} \doteqdot -8\mathrm{i}\,\frac{\kappa_{\widehat{\beta_1\alpha_2}}(2\kappa^2_{\alpha_2}-\kappa^2_{\beta_1})}{\kappa_{\widehat{\alpha_1\alpha_2}}F_{0,\,\alpha_2}}\,\mathrm{e}^{\mathrm{i}\omega\tau}\int_0^\infty u^2\exp\{\mathrm{i}u^2(x/2\kappa_{\alpha_2}-(H-h+z)/2\kappa_{\widehat{\alpha_1\alpha_2}}-H/2\kappa_{\widehat{\beta_1\alpha_2}})\}\,\mathrm{d}u \tag{9·6}
$$

$$
\begin{aligned}
&= 4\sqrt{2}\,\pi^{\frac{1}{2}}\mathrm{i}^{\frac{1}{2}}(\widehat{\alpha_1\alpha_2}/\widehat{\beta_1\alpha_2})\,\alpha_2^5(2/\alpha_2^2-1/\beta_1^2)\,(x\alpha_2-(H-h+z)\,\widehat{\alpha_1\alpha_2}-H\widehat{\beta_1\alpha_2})^{-\frac{3}{2}} \\
&\qquad \times[(Y_1/S_0-Y_0S_1/S_0^2)/F_0]_{\alpha_2}\,\omega^{-\frac{3}{2}}\,\mathrm{e}^{\mathrm{i}\omega\tau},
\end{aligned} \tag{9·7}
$$

where
$$\tau = t-x/\alpha_2-(H-h+z)/\widehat{\alpha_1\alpha_2}-H/\widehat{\beta_1\alpha_2}, \tag{9·8}$$

and at this stage we have extracted the factors ω^4/α_2^4 from F_0, ω^5/α_2^5 from Y_1 and ω^6/α_2^6 from S_0; no confusion should be caused by retaining the same symbols F_0, Y_1, etc., for the reduced expressions, which are now non-dimensional constant coefficients depending only on the properties of the media. In future this procedure will be carried out automatically.

(ii) $\mathscr{R}(\omega)<0$. $\lambda_{\alpha_1}, \lambda_{\beta_2}, \lambda_{\beta_1}$ are all pure imaginary multiples of ω when evaluated at the branch points $\pm\kappa_{\alpha_2}$. Therefore ζ only is associated with a change of sign. The integrand is seen to be even in ζ since Y is odd and S even, so taking account of the fact that the loops Γ_{α_2} and Γ'_{α_2} are described in opposite directions (relative to the location of the points $+\mathrm{i}u$ and $-\mathrm{i}u$) we find that the expressions for ${}_p\phi^{(4)}_{\alpha_2}$ are identical for $\mathscr{R}(\omega)\gtrless 0$. Therefore, we have

$$
{}_p\Phi^{(4)}_{\alpha_2} = 4\sqrt{(2\pi\mathrm{i})}\;\frac{\widehat{\alpha_1\alpha_2}\,\alpha_2^5(2/\alpha_2^2-1/\beta_1^2)}{\widehat{\beta_1\alpha_2}(x\alpha_2-(H-h+z)\widehat{\alpha_1\alpha_2}-H\widehat{\beta_1\alpha_2})^{\frac{3}{2}}}\left[\frac{1}{F_0}\left(\frac{Y_1}{S_0}-\frac{Y_0S_1}{S_0^2}\right)\right]_{\alpha_2}\frac{1}{2\pi\mathrm{i}}\int_\Omega \omega^{-\frac{5}{2}}\,\mathrm{e}^{\mathrm{i}\omega\tau}\,\mathrm{d}\omega \tag{9·9}
$$

$$
\begin{aligned}
&= -\tfrac{16}{3}\sqrt{2}\,(\widehat{\alpha_1\alpha_2}/\widehat{\beta_1\alpha_2})\,\alpha_2^5(2/\alpha_2^2-1/\beta_1^2)\,(x\alpha_2-(H-h+z)\,\widehat{\alpha_1\alpha_2}-H\widehat{\beta_1\alpha_2})^{-\frac{3}{2}} \\
&\qquad \times[(Y_1/S_0-Y_0S_1/S_0^2)/F_0]_{\alpha_2}\tau^{\frac{3}{2}}H(\tau),
\end{aligned} \tag{9·10}
$$

by use of (7·7), which is valid since the contour Ω avoids the origin. Hence, by differentiation, the associated displacements are

$$
\begin{aligned}
{}_pU^{(4)}_{\alpha_2} \doteqdot\; & 8\sqrt{2}\,(\widehat{\alpha_1\alpha_2}/\widehat{\beta_1\alpha_2})\,\alpha_2^4(2/\alpha_2^2-1/\beta_1^2)\,(x\alpha_2-(H-h+z)\,\widehat{\alpha_1\alpha_2}-H\widehat{\beta_1\alpha_2})^{-\frac{3}{2}} \\
& \times[(Y_1/S_0-Y_0S_1/S_0^2)/F_0]_{\alpha_2}\tau^{\frac{1}{2}}H(\tau),
\end{aligned} \tag{9·11}
$$

$$
\begin{aligned}
{}_pW^{(4)}_{\alpha_2} \doteqdot\; & 8\sqrt{2}\,(\alpha_2^5/\widehat{\beta_1\alpha_2})\,(2/\alpha_2^2-1/\beta_1^2)\,(x\alpha_2-(H-h+z)\,\widehat{\alpha_1\alpha_2}-H\widehat{\beta_1\alpha_2})^{-\frac{3}{2}} \\
& \times[(Y_1/S_0-Y_0S_1/S_0^2)/F_0]_{\alpha_2}\tau^{\frac{1}{2}}H(\tau),
\end{aligned} \tag{9·12}
$$

and the velocities
$${}_p\dot{U}^{(4)}_{\alpha_2} = \frac{\partial}{\partial t}({}_pU^{(4)}_{\alpha_2}) \propto \tau^{-\frac{1}{2}}H(\tau), \tag{9·13}$$

$${}_p\dot{W}^{(4)}_{\alpha} = \frac{\partial}{\partial t}({}_pW^{(4)}_{\alpha_2}) \propto \tau^{-\frac{1}{2}}H(\tau). \tag{9·14}$$

Both displacements are seen to have a definite beginning at the instant

$$t = x/\alpha_2 + (H-h+z)/\widehat{\alpha_1\alpha_2} + H/\widehat{\beta_1\alpha_2},$$

that is, just such a time after the initial explosion as would be required to travel from the source to the observer by the path shown in figure 7.

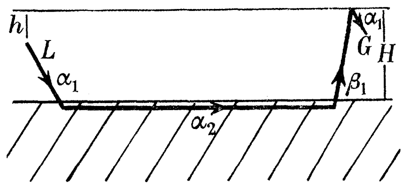

FIGURE 7. Path of pulse ${}_p\phi^{(4)}_{\alpha_2}$.

This is a 'minimum-time path', the disturbance originating as a P-wave in medium I, striking the interface at the critical angle and travelling close to it as a P-wave in medium II, being refracted up as an S-wave in medium I and reaching the observer by reflexion as a P-wave from the surface. On a 'ray theory' such a disturbance could only have zero amplitude, i.e. carry zero energy.

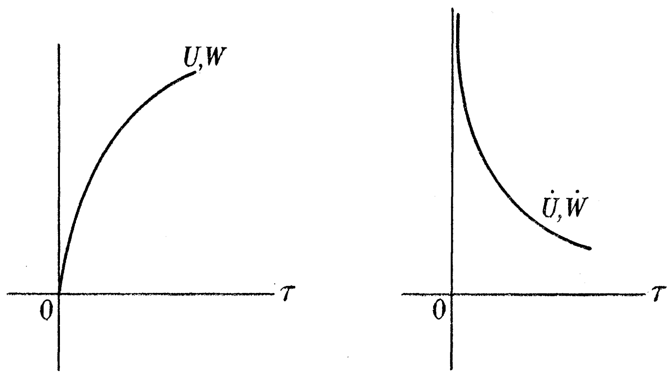

FIGURE 8. Variation of U, W, $\dot{U}$, $\dot{W}$ at the ${}_p\phi^{(4)}_{\alpha_2}$ onset.

Comparing the expressions for ${}_pU^{(4)}_{\alpha_2}$, ${}_pW^{(4)}_{\alpha_2}$ with those for U_0 (the 'direct' pulse), we see that the velocities ${}_p\dot{U}^{(4)}_{\alpha_2}$, ${}_p\dot{W}^{(4)}_{\alpha_2}$ have the same form as the displacements in the direct pulse, that is, the arrival of this 'refracted pulse' is marked by a suddenly acquired great velocity, the initial displacements being zero (figure 8). The amplitude varies with distance approximately as $x^{-\frac{3}{2}}$, while the ratio $U{:}W$ is seen to be $\widehat{\alpha_1\alpha_2}{:}\alpha_2$ or, with our values of constants, approximately 9:8. Our approximation is valid only at the beginning of the motion, but then it is the 'onset' which yields the information on the seismic record. It is reasonable to assume (and the accurate picture of U_0 supports this) that the displacements increase steeply but smoothly to a maximum then subside, and that at a sufficiently great distance from the explosion this component should be felt as a small but sharp pulse.

We may note here that in interpreting a record there is a tendency to select a steep rather than a large but less sharp displacement as marking the arrival of a definite pulse, so that it is changes in $\dot{U}$, $\dot{W}$ rather than in U, W which are sought.

(b) *Contribution from* Γ_{α_1}: ${}_p\phi^{(4)}_{\alpha_1}$

(i) $\mathscr{R}(\omega)>0$. Now on Γ_{α_1} put

$$\lambda_{\alpha_1} = \pm \mathrm{i}u \quad \text{(left-hand and right-hand side of cut).}$$

Then, near κ_{α_1}, we have

$$\zeta = (\kappa^2_{\alpha_1}+\lambda^2_{\alpha_1})^{\frac{1}{2}} \doteqdot \kappa_{\alpha_1} - u^2/2\kappa_{\alpha_1} \doteqdot \kappa_{\alpha_1},$$

$$\zeta\,\mathrm{d}\zeta = -u\,\mathrm{d}u,$$

$$\lambda_{\beta_1} \doteqdot \mathrm{i}(\kappa_{\widehat{\beta_1\alpha_1}} + u^2/2\kappa_{\widehat{\beta_1\alpha_1}}) \doteqdot \mathrm{i}\kappa_{\beta_1\alpha_1}.$$

Write $\qquad Y \doteqdot Y_{0,\,\alpha_1} \pm \mathrm{i}uY_{1,\,\alpha_1} \quad$ (in fact, $Y_{0,\,\alpha_1}\equiv 0$)

$$S \doteqdot S_{0,\,\alpha_1} \pm \mathrm{i}uS_{1,\,\alpha_1}, \quad \text{etc.}$$

Thus

$${}_p\phi^{(4)}_{\alpha_1} = \frac{1}{2}\int_{\Gamma\alpha_1 1} \frac{(-8)\,\zeta\lambda_{\beta_1}(2\zeta^2-\kappa^2_{\beta_1})}{\lambda_{\alpha_1}F}\left(\frac{Y}{S}\right)\exp\{\mathrm{i}\omega t - \mathrm{i}\zeta x - (H-h+z)\,\lambda_{\alpha_1} - H\lambda_{\beta_1}\}\,\mathrm{d}\zeta \qquad (9\cdot15)$$

$$\doteqdot -\frac{1}{2}\int_0^\infty \frac{(-8)\,\mathrm{i}\kappa_{\widehat{\beta_1\alpha_1}}(2\kappa^2_{\alpha_1}-\kappa^2_{\beta_1})}{F_{0,\,\alpha_1}}\left(\frac{Y_1}{S_0}\right)_{\alpha_1}\left[1+\mathrm{i}u\left(-\frac{S_1}{S_0}-\frac{F_1}{F_0}\right)_{\alpha_1}\right]$$

$$\times\exp\{\mathrm{i}\omega(t-x/\alpha_1-H/\widehat{\beta_1\alpha_1}) - \mathrm{i}u(H-h+z) + \mathrm{i}u^2(x/2\kappa_{\alpha_1} - H/2\kappa_{\widehat{\alpha_1\alpha_2}})\}\,(-u)\,\mathrm{d}u$$

$$+\frac{1}{2}\int_0^\infty \quad \text{(same integrand but with } (-\mathrm{i}u) \text{ in place of } (+\mathrm{i}u))\,\mathrm{d}u \qquad (9\cdot16)$$

$$= \frac{-8\kappa_{\widehat{\beta_1\alpha_1}}}{(2\kappa^2_{\alpha_1}-\kappa^2_{\beta_1})}\left(\frac{Y_1}{S_0}\right)_{\alpha_1}\exp\{\mathrm{i}\omega(t-x/\alpha_1-H/(\widehat{\beta_1\alpha_1})\}$$

$$\times\left\{\int_0^\infty u\sin{(H-h+z)}\,u\exp\{\mathrm{i}u^2(x/2\kappa_{\alpha_1}-H/2\kappa_{\widehat{\beta_1\alpha_1}})\}\,\mathrm{d}u\right.$$

$$\left.+\int_0^\infty\left(\frac{S_1}{S_0}+\frac{F_1}{F_0}\right)u^2\cos{(H-h+z)}\,u\exp\{\mathrm{i}u^2(x/2\kappa_{\alpha_1}-H/2\kappa_{\widehat{\beta_1\alpha_1}})\}\,\mathrm{d}u\right\} \qquad (9\cdot17)$$

$$\doteqdot -4\sqrt{2}\,\pi^{\frac{1}{2}}\mathrm{i}^{\frac{3}{2}}\kappa_{\widehat{\beta_1\alpha_1}}(2\kappa^2_{\alpha_1}-\kappa^2_{\beta_1})^{-1}(x/\kappa_{\alpha_1}-H/\kappa_{\widehat{\beta_1\alpha_1}})^{-\frac{3}{2}}(Y_1/S_0)_{\alpha_1}[(H-h+z)+(S_1/S_0+F_1/F_0)_{\alpha_1}]$$

$$\times\exp\{\mathrm{i}\omega[t-x/\alpha_1-H/\widehat{\beta_1\alpha_1}-(H-h+z)^2/2(x\alpha_1-H\widehat{\beta_1\alpha_1})]\}, \qquad (9\cdot18)$$

where we have substituted $\qquad F_{0,\,\alpha_1} = (2\kappa^2_{\alpha_1}-\kappa^2_{\beta_1})^2. \qquad (9\cdot19)$

Now reducing $Y_{1,\,\alpha_1}$, etc., by factors ω^5/α_1^5, etc., we obtain

$${}_p\phi^{(4)}_{\alpha_1} = -4\sqrt{2}\,\pi^{\frac{1}{2}}\,\mathrm{i}^{\frac{3}{2}}\,(\alpha_1/\widehat{\beta_1\alpha_1})\,(x\alpha_1-H\widehat{\beta_1\alpha_1})^{-\frac{3}{2}}\,(2/\alpha_1^2-1/\beta_1^2)^{-1}\,(Y_1/S_0)_{\alpha_1}$$

$$\times\{\omega^{-\frac{1}{2}}(H-h+z)+\omega^{-\frac{3}{2}}\alpha_1(S_1/S_0+F_1/F_0)\}\,\mathrm{e}^{\mathrm{i}\omega\tau}, \qquad (9\cdot20)$$

where $\qquad \tau = t-x/\alpha_1-H/\widehat{\beta_1\alpha_1}-(H-h+z)^2/2(x\alpha_1-H\widehat{\beta_1\alpha_1}), \qquad (9\cdot21)$

and provided $|x\kappa_{\alpha_1}|$ is sufficiently large and $(2H-h+z)/x$ small.

(ii) $\mathscr{R}(\omega)<0$. There are sign changes associated with ζ and λ_{α_2}, so that part of the expression for ${}_p\phi^{(4)}_{\alpha_1}$ has the same sign for $\mathscr{R}(\omega)\gtrless 0$ and part changes sign.

With the notation proposed in (7·14) and (7·15) we find

$${}_p\phi^{(4)}_{\alpha_1} = -4\sqrt{2}\,\pi^{\frac{1}{2}}\,\mathrm{i}^{\frac{3}{2}}(\alpha_1/\widehat{\beta_1\alpha_1})\,(x\alpha_1-H\widehat{\beta_1\alpha_1})^{-\frac{3}{2}}(2/\alpha_1^2-1/\beta_1^2)^{-1}$$

$$\times\{[(Y_1/S_0)_{\alpha_1,\,0}\,\omega^{-\frac{1}{2}}(H-h+z)\,\mathrm{e}^{\mathrm{i}\omega\tau}+(Y_1S_1/S_0^2+Y_1F_1/S_0F_0)_{\alpha_1,\,0}\,\omega^{-\frac{3}{2}}\alpha_1\,\mathrm{e}^{\mathrm{i}\omega\tau}] \qquad (9\cdot22)$$

$$\pm[(Y_1/S_0)_{\alpha_1,\,1}\,\omega^{-\frac{1}{2}}(H-h+z)\,\mathrm{e}^{\mathrm{i}\omega\tau}+(Y_1S_1/S_0^2+Y_1F_1/S_0F_0)_{\alpha_1,\,1}\,\omega^{-\frac{3}{2}}\alpha_1\,\mathrm{e}^{\mathrm{i}\omega\tau}]\} \quad (\mathscr{R}(\omega)\gtrless 0). \qquad (9\cdot23)$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Applying the result (7·6) to the first part of this expression, (9·22), the contribution to ${}_p\Phi^{(4)}_{\alpha_1}$ is

$$-4\sqrt{2}\,\pi^{\frac{1}{2}}\,\mathrm{i}^{\frac{3}{2}}(\alpha_1/\widehat{\beta_1\alpha_1})\,(x\alpha_1-H\widehat{\beta_1\alpha_1})^{-\frac{3}{2}}\,(2/\alpha_1^2-1/\beta_1^2)^{-1}$$
$$\times\{(Y_1/S_0)_{\alpha_1,0}(H-h+z)\,2\,\mathrm{i}^{\frac{1}{2}}\pi^{-\frac{1}{2}}\tau^{\frac{1}{2}}H(\tau)+(Y_1S_1/S_0^2+Y_1F_1/S_0F_0)_{\alpha_1,0}\times\tfrac{4}{3}\alpha_1\,\mathrm{i}^{\frac{3}{2}}\pi^{-\frac{1}{2}}\tau^{\frac{3}{2}}H(\tau)\}, \quad (9·24)$$

and compared with the term in $\tau^{\frac{1}{2}}$ that with a factor $\tau^{\frac{3}{2}}$ may be neglected, since our approximation is only valid for τ very small.

The displacements U and W are derived by direct differentiation. In the same way the second term of (9·23) will give a contribution which is negligible against that from the first. To determine the contribution (9·23) we must consider ${}_p\dot{\phi}^{(4)}_{\alpha_1}$, for the changing form of ${}_p\phi^{(4)}_{\alpha_1}$ across the imaginary axis of ω requires that we use the contour Ω' in the integration. Thus the contribution to ${}_p\dot{\phi}^{(4)}_{\alpha_1}$ is

$$\pm 4\sqrt{2}\,\pi^{\frac{1}{2}}\,\mathrm{i}^{\frac{1}{2}}(\alpha_1/\widehat{\beta_1\alpha_1})\,(x\alpha_1-H\widehat{\beta_1\alpha_1})^{-\frac{3}{2}}\,(2/\alpha_1^2-1/\beta_1^2)^{-1}\,(Y_1/S_0)_{\alpha_1,1}\,(H-h+z)\,\omega^{\frac{1}{2}}\,\mathrm{e}^{\mathrm{i}\omega\tau} \quad (\mathscr{R}(\omega)\gtrless 0), \quad (9·25)$$

and to ${}_p\dot{\Phi}^{(4)}_{\alpha_1}$ is (using (7·8))

$$-4\sqrt{2}\,\mathrm{i}(\alpha_1/\widehat{\beta_1\alpha_1})\,(x\alpha_1-H\widehat{\beta_1\alpha_1})^{-\frac{3}{2}}\,(2/\alpha_1^2-1/\beta_1^2)^{-1}\,(Y_1/S_0)_{\alpha_1,1}\,(H-h+z)\,\tau'^{-\frac{1}{2}}H(\tau'). \quad (9·26)$$

To the first order in x,

$${}_pU^{(4)}_{\alpha_1}\equiv\frac{\partial}{\partial x}\,({}_p\phi^{(4)}_{\alpha_1})\;\doteqdot -\frac{1}{\alpha_1}\,{}_p\dot{\phi}^{(4)}_{\alpha_1}, \quad (9·27)$$

but we can only conveniently determine the time derivative of the vertical displacement for reasons evident from the form of (9·23). Thus adding contributions

$${}_pU^{(4)}_{\alpha_1}\doteqdot -4\sqrt{2}\,(x\alpha_1-H\widehat{\beta_1\alpha_1})^{-\frac{3}{2}}\,\widehat{\beta_1\alpha_1}^{-1}(2/\alpha_1^2-1/\beta_1^2)^{-1}\,(H-h+z)$$
$$\times\{(Y_1/S_0)_{\alpha_1,0}\,\tau^{-\frac{1}{2}}H(\tau)-\mathrm{i}(Y_1/S_0)_{\alpha_1,1}\,\tau'^{-\frac{1}{2}}H(\tau')\}, \quad (9·28)$$

and ${}_pW^{(4)}_{\alpha_1}$ is given by the combined effect of

$$\begin{cases} {}_pW^{(4)}_{\alpha_1(a)}=8\sqrt{2}\,(x\alpha_1-H\widehat{\beta_1\alpha_1})^{-\frac{3}{2}}\,\widehat{\beta_1\alpha_1}^{-1}(2/\alpha_1^2-1/\beta_1^2)^{-1}\,(Y_1/S_0)_{\alpha_1,0} \\ \qquad\times[\tau^{\frac{1}{2}}-\tau^{-\frac{1}{2}}(H-h+z)^2/2(x\alpha_1-H\widehat{\beta_1\alpha_1})]\,H(\tau), \quad (9·29) \\ {}_p\dot{W}^{(4)}_{\alpha_1(b)}=-4\sqrt{2}\,\mathrm{i}(x\alpha_1-H\widehat{\beta_1\alpha_1})^{-\frac{3}{2}}\,\widehat{\beta_1\alpha_1}^{-1}(2/\alpha_1^2-1/\beta_1^2)^{-1}\,(Y_1/S_0)_{\alpha_1,1} \\ \qquad\times[\tau'^{-\frac{1}{2}}-\tau'^{-\frac{3}{2}}(H-h+z)^2/2(x\alpha_1-H\widehat{\beta_1\alpha_1})]\,H(\tau'). \quad (9·30)\end{cases}$$

We may note that the terms neglected in the original approximation (8·16), because they would have introduced further powers of $(x\alpha_1-H\widehat{\beta_1\alpha_1})^{-1}$, would also have integrated to give correspondingly higher powers of τ or τ'. The inclusion of the second terms in (9·29) and (9·30) is therefore significant. Apart from a constant, ${}_pW^{(4)}_{\alpha_1(b)}$ may be obtained by integration of this last relation with respect to τ.

This disturbance does not have a definite beginning but it is concentrated about the instant $t=x/\alpha_1+H/\widehat{\beta_1\alpha_1}+(H-h+z)^2/2(x\alpha_1-H\widehat{\beta_1\alpha_1})$, which is just the time taken to reach the observer by the path shown in figure 9.

This is a minimum-time path predicted by the 'ray theory', the disturbance originating as a P-wave in medium I and reaching the observer by successive reflexions at the surface and interface as an S-wave and a P-wave respectively.

Returning to the expressions for the displacements, the terms governed by $H(\tau')$ provide a 'lead up' to those governed by $H(\tau)$. Both parts of U imply infinite displacements at the instant $t = x/\alpha_1 + H/\widehat{\beta_1\alpha_1} + (H-h+z)^2/2(x\alpha_1 - H\widehat{\beta_1\alpha_1})$, and calculation of $(Y_1/S_0)_{\alpha_1,0}$ and $(Y_1/S_0)_{\alpha_1,1}$ show that the two parts are in opposite phase; the arrival should therefore be felt as a double jerk (figure 10*a*).

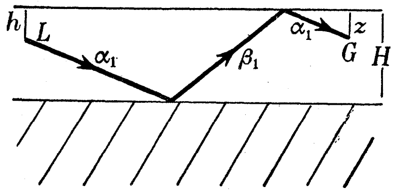

FIGURE 9. Path of pulse ${}_p\phi^{(4)}_{\alpha_1}$.

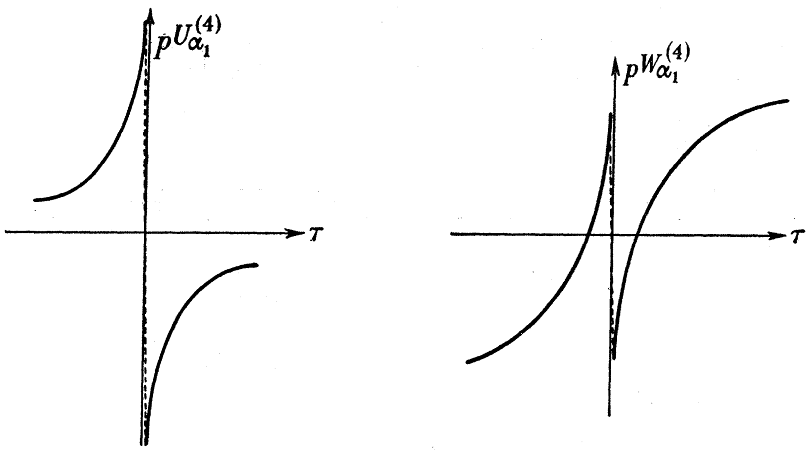

FIGURE 10. ${}_pU^{(4)}_{\alpha_1}$ and ${}_pW^{(4)}_{\alpha_1}$ as functions of τ.

The failure of the Hankel function to represent physical conditions by producing non-finite jerks is particularly inconvenient when we attempt to obtain a picture of ${}_pW^{(4)}_{\alpha_1}$. The additional power of $1/x$ in the second terms of (9·30) and (9·31) would make them insignificant for x large (as postulated), but for the fact that at the instant $\tau = 0$ the factors $\tau'^{-\frac{1}{2}}$, $\tau^{-\frac{1}{2}}$ give infinities while the other terms in $\tau'^{\frac{1}{2}}$, $\tau^{\frac{1}{2}}$ are zero. We shall interpret this as in figure 10, the $\tau'^{-\frac{1}{2}}$, $\tau^{-\frac{1}{2}}$ terms contributing slight but sharp jerks at the instant $\tau = 0$, but otherwise the shape of the pulse being determined by the $\tau'^{\frac{1}{2}}$, $\tau^{\frac{1}{2}}$ factors. Thus W, the vertical displacement, should appear as a brief oscillation modified by a slight jerk at the instant $t = x/\alpha_1 + H/\widehat{\beta_1\alpha_1} + (H-h+z)^2/2(x\alpha_1 - H\widehat{\beta_1\alpha_1})$, this jerk decreasing in relative importance with increasing distance from the epicentre.

(*c*) *Contribution from* Γ_{β_2} : ${}_p\phi^{(4)}_{\beta_2}$

(i) $\mathscr{R}(\omega) > 0$. Now on Γ_{β_2} put

$$\lambda_{\beta_2} = \pm \mathrm{i}u \quad \text{(left-hand and right-hand side of cut).}$$

Then, near κ_{β_2},

$$\zeta = (\kappa^2_{\beta_2} - u^2)^{\frac{1}{2}} \doteqdot \kappa_{\beta_2} - u^2/2\kappa_{\beta_2} \doteqdot \kappa_{\beta_2},$$

$$\zeta\,\mathrm{d}\zeta = -u\,\mathrm{d}u,$$

$$\lambda_{\alpha_1} = (\zeta^2 - \kappa^2_{\alpha_1})^{\frac{1}{2}} \doteqdot \kappa_{\widehat{\beta_2\alpha_1}}(1 - u^2/2\kappa_{\widehat{\beta_2\alpha_1}}) \doteqdot \kappa_{\widehat{\beta_2\alpha_1}},$$

$$\lambda_{\beta_1} = (\zeta^2 - \kappa^2_{\beta_1})^{\frac{1}{2}} \doteqdot \mathrm{i}\kappa_{\widehat{\beta_1\beta_2}}(1 + u^2/2\kappa_{\widehat{\beta_1\beta_2}}) \doteqdot \mathrm{i}\kappa_{\widehat{\beta_1\beta_2}}.$$

Write $$Y \doteqdot Y_{0,\beta_2} \pm \mathrm{i}uY_{1,\beta_2}, \quad \text{etc.} \quad (\text{note: } F_{1,\beta_2} \equiv 0).$$
Thus

$$
{}_p\phi^{(4)}_{\beta_2} = \frac{1}{2}\int_{\Gamma\beta_2} \frac{-8\zeta\lambda_{\beta_1}(2\zeta^2-\kappa^2_{\beta_1})}{\lambda_{\alpha_1}F}\left(\frac{Y}{S}\right)\exp\{\mathrm{i}\omega t - \mathrm{i}\zeta x-(H-h+z)\,\lambda_{\alpha_1}-H\lambda_{\beta_1}\}\,\mathrm{d}\zeta
$$

$$
\doteqdot \left\{-\frac{1}{2}\int_0^\infty \frac{-8\mathrm{i}\kappa_{\widehat{\beta_1\beta_2}}(2\kappa^2_{\beta_2}-\kappa^2_{\beta_1})\,\mathrm{i}u}{\kappa_{\widehat{\beta_2\alpha_1}}F_{0,\beta_2}}\left(\frac{Y_1}{S_0}-\frac{Y_0S_1}{S_0^2}\right)_{\beta_2}\times(-u)\exp\{\mathrm{i}u^2(x/2\kappa_{\beta_2}-H/2\kappa_{\widehat{\beta_1\beta_2}})\}\,\mathrm{d}u\right.
$$

$$
\left.+\frac{1}{2}\int_0^\infty \quad (\text{same integrand but with } (-\mathrm{i}u) \text{ in place of } (+\mathrm{i}u))\,\mathrm{d}u\right\}
$$

$$
\times\exp\{\mathrm{i}\omega t-\mathrm{i}\zeta x-(H-h+z)\,\lambda_{\alpha_1}-H\lambda_{\beta_1}\} \qquad (9\cdot31)
$$

(the lower-order terms cancelling on the two sides of the cut)

$$
= \frac{8\kappa_{\widehat{\beta_1\beta_2}}(2\kappa^2_{\beta_2}-\kappa^2_{\beta_1})}{\kappa_{\widehat{\beta_2\alpha_1}}F_{0,\beta_2}}\left(\frac{Y_1}{S_0}-\frac{Y_0S_1}{S_0^2}\right)_{\beta_2}\exp\{\mathrm{i}\omega(t-x/\beta_2-H/\widehat{\beta_1\beta_2})-(H-h+z)\,\kappa_{\widehat{\beta_2\alpha_1}}\}
$$

$$
\times\int_0^\infty u^2\exp\{\mathrm{i}u^2(x/2\kappa_{\beta_2}-H/2\kappa_{\widehat{\beta_1\beta_2}})\}\,\mathrm{d}u
$$

$$
= 8\kappa_{\widehat{\beta_1\beta_2}}(2\kappa^2_{\beta_2}-\kappa^2_{\beta_1})\,(\kappa_{\widehat{\beta_2\alpha_1}}F_{0,\beta_2})^{-1}\,(Y_1/S_0-Y_0S_1/S_0^2)_{\beta_2}
$$

$$
\times\exp\{\mathrm{i}\omega(t-x/\beta_2-H/\widehat{\beta_1\beta_2})-(H-h+z)\,\kappa_{\widehat{\beta_2\alpha_1}}\}\,\mathrm{i}^{\frac{3}{2}}\sqrt{(\tfrac{1}{2}\pi)}\,(x/\kappa_{\beta_2}-H/\kappa_{\widehat{\beta_1\beta_2}})^{-\frac{3}{2}}, \qquad (9\cdot32)
$$

and reducing Y_{1,β_2}, etc., by factors ω^5/β_2^5, etc., we obtain

$$
{}_p\phi^{(4)}_{\beta_2} = 4\sqrt{2}\,\pi^{\frac{1}{2}}\,\mathrm{i}^{\frac{3}{2}}\,(2/\beta_2^2-1/\beta_1^2)\,\widehat{\beta_2\alpha_1}\beta_2^5\widehat{\beta_1\beta_2}^{-1}(x\beta_2-H\widehat{\beta_1\beta_2})^{-\frac{3}{2}}\omega^{-\frac{3}{2}}\,(Y_1/F_0S_0-Y_0S_1/F_0S_0^2)_{\beta_2}\,\mathrm{e}^{-p\omega+\mathrm{i}\omega\tau}, \qquad (9\cdot33)
$$

where

$$
\left.\begin{aligned} \tau &= t-x/\beta_2-H/\widehat{\beta_1\beta_2},\\ p &= (H-h+z)/\widehat{\beta_2\alpha_1}.\end{aligned}\right\} \qquad (9\cdot34)
$$

(ii) $\mathscr{R}(\omega)<0$. There are sign changes associated with ζ, λ_{α_2} and λ_{α_1}, so that with the notation of (7·14) we find

$$
{}_p\phi^{(4)}_{\beta_2} = 4\sqrt{2}\,\pi^{\frac{1}{2}}\,\mathrm{i}^{\frac{3}{2}}\,(2/\beta_2^2-1/\beta_1^2)\,\widehat{\beta_2\alpha_1}\beta_2^5\widehat{\beta_1\beta_2}^{-1}(x\beta_2-H\widehat{\beta_1\beta_2})^{-\frac{3}{2}}\,\omega^{-\frac{3}{2}}
$$

$$
\times\{(Y_1/F_0S_0-Y_0S_1/F_0S_0^2)_{\beta_2,1}\,\mathrm{e}^{\mp p\omega+\mathrm{i}\omega\tau}\pm(Y_1/F_0S_0-Y_0S_1/F_0S_0^2)_{\beta_2,0}\,\mathrm{e}^{\mp p\omega+\mathrm{i}\omega\tau}\}. \qquad (9\cdot35)
$$

The form of either part of this expression means that to determine the contribution corresponding to the initial unit pulse we must consider ${}_p\dot{u}^{(4)}_{\beta_2}$, say, and derive ${}_p\dot{U}^{(4)}_{\beta_2}$ by integration on Ω':

$$
{}_p\dot{U}^{(4)}_{\beta_2} = \frac{1}{2\pi\mathrm{i}}\int_{\Omega'}{}_p\dot{u}^{(4)}_{\beta_2}\frac{\mathrm{d}\omega}{\omega} = \frac{1}{2\pi\mathrm{i}}\int\frac{\omega}{\beta_2}\,{}_p\dot{\phi}^{(4)}_{\beta_2}\,\mathrm{d}\omega,
$$

and using the results (7·11) and (7·12) we find

$$
{}_p\dot{U}^{(4)}_{\beta_2} = 4\sqrt{2}\,\widehat{\beta_2\alpha_1}^{\frac{3}{2}}\,\widehat{\beta_1\beta_2}^{-1}\,\beta_2^4(2/\beta_2^2-1/\beta_1^2)\,(x\beta_2-H\widehat{\beta_1\beta_2})^{-\frac{3}{2}}\,(H-h+z)^{-\frac{1}{2}}
$$

$$
\times\cos^{\frac{1}{2}}\psi\{-(Y_1/F_0S_0-Y_0S_1/F_0S_0^2)_{\beta_2,0}\sin(\tfrac{1}{2}\psi-\tfrac{1}{4}\pi)+\mathrm{i}(Y_1/F_0S_0-Y_0S_1/F_0S_0^2)_{\beta_2,1}\sin(\tfrac{1}{2}\psi+\tfrac{1}{4}\pi)\},
$$

where

$$
\tan\psi = \tau/p. \qquad (9\cdot36)
$$

Similarly

$$
{}_p\dot{W}^{(4)}_{\beta_2} = 4\sqrt{2}\,\beta_2^5\,\widehat{\beta_2\alpha_1}^{\frac{1}{2}}\,\widehat{\beta_1\beta_2}^{-1}(2/\beta_2^2-1/\beta_1^2)\,(x\beta_2-H\widehat{\beta_1\beta_2})^{-\frac{3}{2}}\,(H-h+z)^{-\frac{1}{2}}
$$

$$
\times\cos^{\frac{1}{2}}\psi\{\mathrm{i}(Y_1/F_0S_0-Y_0S_1/F_0S_0^2)_{\beta_2,1}\sin(\tfrac{1}{2}\psi-\tfrac{1}{4}\pi)+(Y_1/F_0S_0-Y_0S_1/F_0S_0^2)_{\beta_2,0}\sin(\tfrac{1}{2}\psi+\tfrac{1}{4}\pi)\}. \qquad (9\cdot37)
$$

The function $$\cos^{\frac{1}{2}}\psi\,\sin\left(\tfrac{1}{2}\psi+\tfrac{1}{4}\pi\right),\quad -\tfrac{1}{2}\pi\leqslant\psi\leqslant\tfrac{1}{2}\pi,$$

is shown in figure 11*a* as a function of τ. It is easily verified that it has a maximum at $\psi=\frac{1}{6}\pi$ corresponding to

$$\tau=t-x/\beta_2-H/\widehat{\beta_1\beta_2}=(H-h+z)/\sqrt{3}\,\widehat{\beta_2\alpha_1}$$

(i.e. same for all x) and decreases steeply towards negative τ and more gradually towards large positive τ, in both directions to the value zero. Also, the function

$$\cos^{\frac{1}{2}}\psi\,\sin\left(\tfrac{1}{2}\psi-\tfrac{1}{4}\pi\right)$$

may be written $$-\cos^{\frac{1}{2}}\psi\,\sin\left(-\tfrac{1}{2}\psi+\tfrac{1}{4}\pi\right),$$

and its graph is therefore derived from figure 11(*a*) by successive reflexions in the two axes (see figure 11(*b*)).

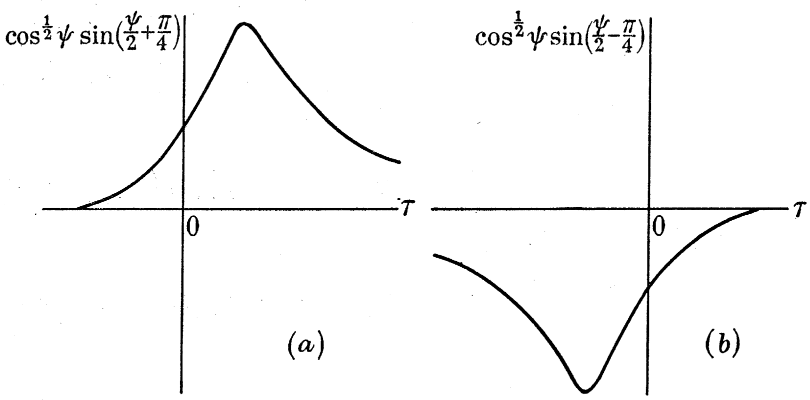

FIGURE 11. (*a*) the function $\cos^{\frac{1}{2}}\psi\,\sin(\frac{1}{2}\psi+\frac{1}{4}\pi)$, (*b*) the function $\cos^{\frac{1}{2}}\psi\,\sin(\frac{1}{2}\psi-\frac{1}{4}\pi)$.

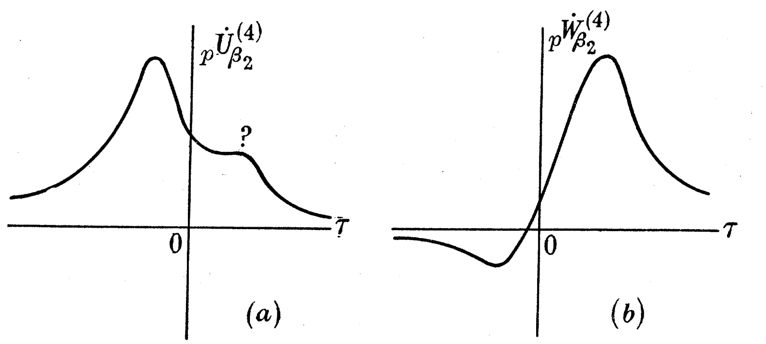

FIGURE 12. (*a*) ${}_p\dot{U}^{(4)}_{\beta_2}$, (*b*) ${}_p\dot{W}^{(4)}_{\beta_2}$, as functions of τ.

Each of $\dot{U}$ and $\dot{W}$ is given then by the algebraic sum of multiples of these two functions. It is evident from the relevant factors in (9·36) and (9·37) that if the two parts of $\dot{U}$ are in phase the two parts of $\dot{W}$ are in opposite phase and vice versa. Calculation shows that the ratio

$$(Y_1/F_0S_0-Y_0S_1/F_0S_0^2)_{\beta_2,0}:\mathrm{i}(Y_1/F_0S_0-Y_0S_1/F_0S_0^2)_{\beta_2,1}$$

is approximately equal to 5:2. The displacements should therefore vary roughly as in figures 12 (*a*) and (*b*), $\dot{U}$ dominated by the $\cos^{\frac{1}{2}}\psi\,\sin(\frac{1}{2}\psi-\frac{1}{4}\pi)$ function and $\dot{W}$ by the other. Validity of the results will depend on the smallness of $\tau\beta_2/x$, so that we may expect the estimated value and slope of $\dot{U}$, $\dot{W}$ at $\tau=0$ to be good and the location of the peak to increase in accuracy with distance x.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

(d) Contribution from Γ_{β_1} : ${}_p\phi^{(4)}_{\beta_1}$

(i) $\mathscr{R}(\omega) > 0$. Now on Γ_{β_1} put

$$\lambda_{\beta_1} = \pm \mathrm{i}u \quad \text{(left-hand and right-hand side of cut).}$$

Then, near κ_{β_1},
$$\zeta = (\kappa^2_{\beta_1} - u^2)^{\frac{1}{2}} \doteqdot \kappa_{\beta_1} - u^2/2\kappa_{\beta_1} \doteqdot \kappa_{\beta_1},$$
$$\zeta\,\mathrm{d}\zeta = -u\,\mathrm{d}u,$$
$$\lambda_{\alpha_1} = (\zeta^2 - \kappa^2_{\alpha_1})^{\frac{1}{2}} \doteqdot \kappa_{\widehat{\beta_1\alpha_1}}.$$

Write
$$Y \doteqdot Y_{0,\beta_1} \pm \mathrm{i}uY_{1,\beta_1}, \quad \text{etc.} \quad (\text{note: } Y_{1,\beta_1} \equiv 0).$$

Thus

$$\begin{aligned} {}_p\phi^{(4)}_{\beta_1} &= \frac{1}{2}\int_{\Gamma\beta_1} \frac{(-8)\,\zeta\lambda_{\beta_1}(2\zeta^2-\kappa^2_{\beta_1})}{\lambda_{\alpha_1}F}\left(\frac{Y}{S}\right)\exp\{\mathrm{i}\omega t - \mathrm{i}\zeta x - (H-h+z)\,\lambda_{\alpha_1} - H\lambda_{\beta_1}\}\,\mathrm{d}\zeta \\ &\doteqdot -\frac{1}{2}\int_0^\infty \frac{(-8)\,\kappa^2_{\beta_2}}{\kappa_{\widehat{\beta_1\alpha_1}}F_{0,\beta_1}}\left(\frac{Y_0}{S_0}\right)_{\beta_1} \mathrm{i}u\exp\{\mathrm{i}\omega t - \mathrm{i}x\kappa_{\beta_1} + \mathrm{i}u^2x/2\kappa_{\beta_1} - \mathrm{i}uH - (H-h+z)\,\kappa_{\widehat{\beta_1\alpha_1}}\}\,(-u)\,\mathrm{d}u \\ &\quad + \frac{1}{2}\int_0^\infty \text{(same integrand but with } (-\mathrm{i}u) \text{ in place of } (+\mathrm{i}u)\,\mathrm{d}u \qquad (9\cdot38) \\ &= \frac{-8\mathrm{i}\kappa^2_{\beta_1}}{\kappa_{\widehat{\beta_1\alpha_1}}F_{0,\beta_1}}\left(\frac{Y_0}{S_0}\right)_{\beta_1}\exp\{\mathrm{i}\omega(t-x/\beta_1) - (H-h+z)\,\kappa_{\widehat{\beta_1\alpha_1}}\}\int_0^\infty u^2\cos Hu\exp\{\mathrm{i}u^2x/2\kappa_{\beta_1}\}\,\mathrm{d}u \\ &= 4\sqrt{2}\,\pi^{\frac{1}{2}}\,\mathrm{i}^{\frac{1}{2}}\,\kappa^{\frac{3}{2}}_{\beta_1}\kappa^{-1}_{\widehat{\beta_1\alpha_1}}(Y_0/F_0S_0)_{\beta_1}\,x^{-\frac{3}{2}}\exp\{\mathrm{i}\omega(t-x/\beta_1-H^2/2x\beta_1) - (H-h+z)\,\kappa_{\widehat{\beta_1\alpha_1}}\}. \qquad (9\cdot39) \end{aligned}$$

Substituting
$$F_{0,\beta_1} = \kappa^4_{\beta_1}$$

and reducing Y_{0,β_1}, etc., by factors ω^6/β^6_1, etc., we obtain

$$ {}_p\phi^{(4)}_{\beta_1} = 4\,\sqrt{2}\,\pi^{\frac{1}{2}}\,\mathrm{i}^{\frac{1}{2}}\,\widehat{\beta_1\alpha_1}\,\beta^{\frac{1}{2}}_1\,x^{-\frac{3}{2}}(Y_0/S_0)_{\beta_1}\,\omega^{-\frac{3}{2}}\,\mathrm{e}^{-p\omega+\mathrm{i}\omega\tau}, \qquad (9\cdot40)$$

where
$$\left.\begin{aligned} p &= (H-h+z)/\widehat{\beta_1\alpha_1}, \\ \tau &= t - x/\beta_1 - H^2/2x\beta_1. \end{aligned}\right\} \qquad (9\cdot41)$$

(ii) $\mathscr{R}(\omega) < 0$. There are sign changes associated with ζ, λ_{α_2}, λ_{α_1}, λ_{β_2}, but we find, with the notation of (7·14), that
$$(Y_0/S_0)_{\beta_1,0} \equiv 0, \qquad (9\cdot42)$$
and therefore we have

$$ {}_p\phi^{(4)}_{\beta_1} = 4\sqrt{2}\,\pi^{\frac{1}{2}}\,\mathrm{i}^{\frac{1}{2}}\,\widehat{\beta_1\alpha_1}\,\beta^{\frac{1}{2}}_1\,x^{-\frac{3}{2}}(Y_0/S_0)_{\beta_1,1}\,\omega^{-\frac{3}{2}}\,\mathrm{e}^{\mp p\omega+\mathrm{i}\omega\tau} \quad (\mathscr{R}(\omega) \gtrless 0). \qquad (9\cdot43)$$

Again, in order to obtain the response to an initial unit pulse, we are obliged (by the occurrence of the factor $\omega^{-\frac{3}{2}}$ in the above) to consider ${}_p\dot{u}^{(4)}_{\beta_1}$, ${}_p\dot{w}^{(4)}_{\beta_1}$. Then

$$ {}_p\dot{U}^{(4)}_{\beta_1} = \frac{1}{2\pi\mathrm{i}}\int_{\Omega'} {}_p\dot{u}^{(4)}_{\beta_1}\frac{\mathrm{d}\omega}{\omega} = \frac{1}{2\pi\mathrm{i}}\int_{\Omega'}(\omega/\beta_1)\,{}_p\phi^{(4)}_{\beta_1}\,\mathrm{d}\omega, \qquad (9\cdot44)$$

and
$$ {}_p\dot{W}^{(4)}_{\beta_1} = \frac{1}{2\pi\mathrm{i}}\int_{\Omega'} {}_p\dot{w}^{(4)}_{\beta_1}\frac{\mathrm{d}\omega}{\omega} = \frac{1}{2\pi\mathrm{i}}\int_{\Omega'}(\mp\mathrm{i}\omega/\widehat{\beta_1\alpha_1})\,{}_p\phi^{(4)}_{\beta_1}\,\mathrm{d}\omega \quad (\mathscr{R}(\omega) \gtrless 0), \qquad (9\cdot45)$$

and using the results (7·11) and (7·12) we arrive at

$$ {}_p\dot{U}^{(4)}_{\beta_1} = 4\sqrt{2}\,\beta^{-\frac{1}{2}}_1\,\widehat{\beta_1\alpha_1}^{\frac{3}{2}}\,x^{-\frac{3}{2}}(H-h+z)^{-\frac{1}{2}}\,(Y_0/S_0)_{\beta_1,1}\cos^{\frac{1}{2}}\psi\,\sin\left(\tfrac{1}{2}\psi + \tfrac{1}{4}\pi\right), \qquad (9\cdot46)$$

$$ {}_p\dot{W}^{(4)}_{\beta_1} = 4\sqrt{2}\,\beta^{\frac{1}{2}}_1\,\widehat{\beta_1\alpha_1}^{\frac{1}{2}}\,x^{-\frac{3}{2}}(H-h+z)^{-\frac{1}{2}}\,(Y_0/S_0)_{\beta_1,1}\cos^{\frac{3}{2}}\psi\,\sin\left(\tfrac{1}{2}\psi - \tfrac{1}{4}\pi\right), \qquad (9\cdot47)$$

where $$\tan\psi = \tau/p, \tag{9·48}$$

valid so long as x is sufficiently great compared with H, h, z and $\beta_1\tau/x$ is small.

We see that these expressions are simpler than those contributed by Γ_{β_2} although of the same type. Their variations are described by figures 11 (*a*) and (*b*), and the ratio of the maximum velocities $\dot{U}_{\max.} : \dot{W}_{\max.}$ (occurring before and after the instant $t = x/\beta_1 + H^2/2x\beta_1$ respectively) is given by the ratio $\widehat{\beta_1\alpha_1} : \beta_1$, i.e. $\surd 3 : \surd 2$.

(*e*) *The contribution from* Γ_{γ_1}: ${}_p\phi^{(4)}_{\gamma_1}$

(i) $\mathscr{R}(\omega) > 0$.

$${}_p\phi^{(4)}_{\gamma_1} = \frac{1}{2}\int_{\Gamma_{\gamma_1}} \frac{-8\zeta\lambda_{\beta_1}(2\zeta^2-\kappa^2_{\beta_1})}{\lambda_{\alpha_1}F}\left(\frac{Y}{S}\right)\exp\{\mathrm{i}\omega t - \mathrm{i}\zeta x - (H-h+z)\,\lambda_{\alpha_1} - H\lambda_{\beta_1}\}\,\mathrm{d}\zeta$$

$$= -2\pi\mathrm{i}\times\text{residue at }\kappa_{\gamma_1} \tag{9·49}$$

$$= \{8\pi\mathrm{i}\,\kappa_{\gamma_1}\kappa_{\widehat{\gamma_1\beta_1}}(2\kappa^2_{\gamma_1}-\kappa^2_{\beta_1})/\kappa_{\widehat{\gamma_1\alpha_1}}F'_{\gamma_1}\}\,(Y/S)_{\gamma_1}\exp\{\mathrm{i}\omega(t-x/\gamma_1) - H\kappa_{\widehat{\gamma_1\beta_1}} - (H-h+z)\,\kappa_{\widehat{\gamma_1\alpha_1}}\}$$

$$= A_r(Y/S)_{\gamma_1}\,\mathrm{e}^{\mathrm{i}\omega\tau-\omega p}, \tag{9·50}$$

where
$$\left.\begin{aligned} F' &= \left(\frac{\partial F}{\partial\zeta}\right)_\omega, \\ \tau &= t - x/\gamma_1, \\ p &= (H-h+z)/\widehat{\gamma_1\alpha_1} - H/\widehat{\gamma_1\beta_1}, \end{aligned}\right\} \tag{9·51}$$

$$A_r = 2\pi\mathrm{i}(2/\gamma_1^2 - 1/\beta_1^2)/\widehat{\gamma_1\beta_1}^2[2(2/\gamma_1^2-1/\beta_1^2)/\widehat{\gamma_1\alpha_1}\,\widehat{\gamma_1\beta_1} - (2/\gamma_1^2 - 1/\widehat{\gamma_1\alpha_1}^2 - 1/\widehat{\gamma_1\beta_1}^2)/\gamma_1^2].$$

(ii) $\mathscr{R}(\omega) < 0$. Apart from the change of sign due to the odd power of ζ, $(Y/S)_{\gamma_1}$ changes sign ($\mathscr{R}(\omega) \gtrless 0$) due to the change in λ_{α_2}, λ_{α_1}, λ_{β_2}, λ_{β_1}, i.e.

$$(Y/S)_{\gamma_1} = \pm(Y/S)_{\gamma_1,1} \quad (\mathscr{R}(\omega) \gtrless 0),$$
$$(Y/S)_{\gamma_1,0} \equiv 0. \tag{9·52}$$

We therefore find that $${}_p\phi^{(4)}_{\gamma_1} = \pm A_r(Y/S)_{\gamma_1,1}\,\mathrm{e}^{\mathrm{i}\omega\tau\mp\omega p} \quad (\mathscr{R}(\omega) \gtrless 0). \tag{9·53}$$

These expressions are exact. Their form is such that, to derive the response to the unit pulse, we must avoid the sector $\omega = -\frac{1}{2}\pi \pm \epsilon$ (ϵ small) in the usual way by choosing as ω contour the curve Ω' through the origin and considering ${}_pu^{(4)}_{\gamma_1}$ and ${}_pw^{(4)}_{\gamma_1}$. We then find that

$$\left.\begin{aligned} {}_pU^{(4)}_{\gamma_1} &= \frac{A_r(Y/S)_{\gamma_1,1}}{\pi\gamma_1}\frac{\tau}{p^2+\tau^2}, \\ {}_pW^{(4)}_{\gamma_1} &= \frac{-A_r(Y/S)_{\gamma_1,1}}{\pi\,\widehat{\gamma_1\alpha_1}}\frac{p}{p^2+\tau^2}. \end{aligned}\right\} \tag{9·54}$$

The variation of U, W with τ is shown in figure 13, the stationary values of U occurring at $\tau = p$ and the maximum displacements, horizontal and vertical, being proportional to $1/p$, i.e. $((H-h+z)/\widehat{\gamma_1\alpha_1} + H/\widehat{\gamma_1\beta_1})^{-1}$. The disturbance is of the nature of a pulse travelling over the surface without loss of amplitude and with velocity γ_1. This particular 'Rayleigh component' is greatest at the surface and for a source situated near the interface. It is, of course, defined within the surface layer only.

Similar determinations were made of each of the zero-order and first-order contributions, and expressions for the associated disturbances are given in appendix 2. Where two or more expressions are bracketed together it is implied that the total component disturbance is given by the combined effect of these. There is a resemblance between groups of terms (see (5·15) to (5·18)) which reduces the total of distinct integrations, and a certain reciprocity between ${}_p\phi$ and ${}_s\psi$ and between ${}_s\phi$ and ${}_p\psi$.

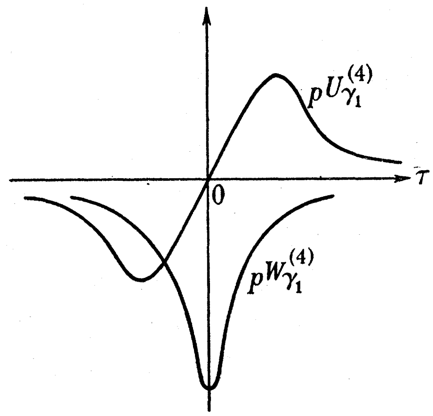

FIGURE 13. ${}_pU^{(4)}_{\gamma_1}$ and ${}_pW^{(4)}_{\gamma_1}$ from ${}_p\phi^{(4)}$ on Γ_{γ_1}.

10. General discussion of the pulse representation

From the illustration of ${}_p\phi^{(4)}$ alone we may predict the nature of any component to the following extent. A general term of the series into which the original ϕ, ψ integrands were expanded contains an exponential of the form

$$\exp\{i\omega t - i\zeta x - h_1\lambda_{\alpha_1} - h_2\lambda_{\beta_1}\}. \tag{10·1}$$

For large x, the main contributions to the integrals come from the immediate neighbourhood of the branch-points and the pole. If, on substitution at any one of these, both λ_{α_1} and λ_{β_1} are to a first approximation pure imaginary multiples of ω, then both the h_1 and h_2 terms appear in the time factor; this time factor then describes a complete travel-path from source to observer and, moreover, a minimum-time path, and the associated disturbance will be sharp like the generating pulse. Further, if we integrate near $\zeta = \omega/c$, say, and x is large, the time factor is dominated by the part $t - x/c$, and we deduce that the disturbance travelled most of the way with velocity c.

On the other hand, if either (or both) of λ_{α_1}, λ_{β_1} becomes a real multiple of ω then the corresponding term does not appear in the time factor but in the amplitude factor as a damping effect. We can no longer associate a travel path with the disturbance, which is now a blunt pulse and must represent part of the diffraction which the ray theory ignores.

Thus, from the loops Γ_{α_2}, Γ_{α_1} we may expect entirely minimum-time path pulses. From the loops Γ_{β_2}, Γ_{β_1} we shall derive minimum-time path pulses only when $h_1 = 0$; otherwise the pulses will be of the second type, blunt and with no associated travel path. All this is borne out by the actual determination of the contributions. We may summarize the results as follows:

(1) The zero-order terms represent the generating pulse and such consequent motion as is due to the presence of the free surface. They provide such motion as would occur if

the surface layer was of very great depth and the second medium absent, that is, the solution obtained by Lapwood for the line source in a uniform semi-infinite medium.

(2) The first and higher-order terms represent the modification due to the finite depth of the layer and the presence of the underlying medium. Each gives five distinct contributions which may be classified according to the part of the contour from which they are derived.

(*a*) *The loop* Γ_{α_2}. This provides all the minimum-time path pulses which appear to have travelled in the lower medium, along the interface, with compressional wave-velocity α_2 after refraction at the critical angle. In the surface layer the motion may be distortional or compressional (see figure 14).

These are sharp pulses, the displacements rising steeply but smoothly from zero at the predicted arrival time. It is the rate of change of displacement which shows a 'jerk' at the onset like the generating disturbance. The pulses vary in intensity as $x^{-\frac{3}{2}}$.

(*b*) *The loop* Γ_{α_1}. This provides minimum-time path pulses corresponding to travel entirely within the layer. The greatest part of the path is travelled with velocity α_1; the motion is therefore purely compressional or else involves a change of type. When h_1 is zero, propagation is mainly over the surface as a compressional wave but beginning and ending as a distortional wave within the layer. This is not, however, a 'surface wave' in the ordinary sense. The surface travel here corresponds to reflexion at 'grazing angle', and its existence may be deduced from the ray theory.

Each of these pulses is compounded of two parts, a disturbance which culminates at $\tau = 0$ (the ray-path arrival time) and a disturbance entirely subsequent to that time—a sort of mirror image of the first about $\tau = 0$. ${}_p\phi$- and ${}_s\phi$-terms give rise to horizontal displacements like the initial pulse, that is, sudden jerks, and if the 'fore' and 'after' parts have opposite sign the result should be a double jerk. The vertical displacements are essentially continuous and may be expected to appear as smooth steep swings but modified by a slight jerk which decreases in relative importance as x increases. The ${}_p\psi$ terms give rise to both horizontal and vertical displacements of the same type as the generating pulse. The ${}_s\psi$ terms give continuous displacements; it is in the 'rate of change' of these that a jerk is felt.

When propagation is partly distortional and partly compressional the displacements U, W die out with distance as $x^{-\frac{3}{2}}$. When wholly compressional, the vertical varies as $x^{-\frac{3}{2}}$ but the horizontal as $x^{-\frac{1}{2}}$. At the surface, however, the various components of U are seen to annihilate one another and next approximations would show a decrease as $x^{-\frac{3}{2}}$.

(*c*) *The loop* Γ_{β_2}. The nature of the Γ_{β_2} contributions depends on the form of the original exponent. If it contains no λ_{α_1} term, i.e. $h_1 = 0$, then we derive refracted minimum-time path pulses as from Γ_{α_1}, but now travel along the interface in the lower medium is with distortional velocity β_2 and through the layer it is with distortional velocity β_1 (an essential consequence of Snell's law with our choice of $\alpha_2 > \alpha_1 > \beta_2 > \beta_1$). These, together with the Γ_{α_2} contributions, provide the complete set of minimum-time path disturbances involving critical refraction at the interface.

The remainder of contributions cannot be associated with 'paths'; their time factors are 'incomplete' in that they do not correspond to travel from the source to the observer. They are diffraction phenomena, blunt pulses which we are able to describe in terms of $\dot{U}$ and $\dot{W}$.

Each is a combination of two forms shown in figure 11 and is centred about $t = x/\beta_2$ or some later time depending on the relation of h_1, h_2 to H, h and z. When $h_1 = z$, a contribution which is everywhere else of this type becomes at the surface part of a minimum-time path disturbance.

(*d*) *The loop* Γ_{β_1}. Again, if $h_1 = 0$ the contributions are minimum-time path pulses, propagation is purely distortional, is confined to the layer and involves successive reflexions at the surface and interface. The displacements show the same time and distance variations as the purely compressional disturbances from Γ_{α_1}, except that the descriptions of U and W are now interchanged.

Otherwise, contributions represent further diffraction effects due to the curvature of the wave fronts. They are rather simpler in form than the corresponding contributions from Γ_{β_2}; either $\dot{U}$ varies as in figure 11*a* and $\dot{W}$ as in figure 11*b* or vice versa.

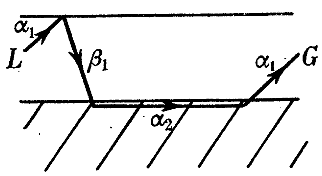

FIGURE 14. Typical Γ_{α_2}-ray path.

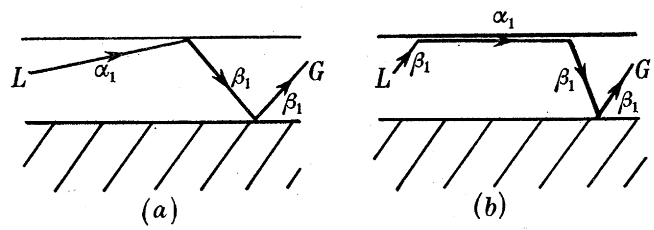

FIGURE 15. Typical Γ_{α_1}-ray path. (*a*) $h_1 \neq 0$; (*b*) $h_1 = 0$.

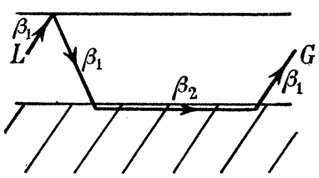

FIGURE 16. Typical Γ_{β_2}-ray path, $h_1 = 0$.

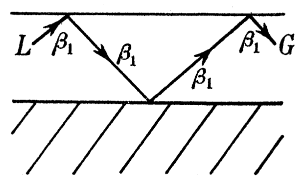

FIGURE 17. Typical Γ_{β_1}-ray path, $h_1 = 0$.

(*e*) All contributions from Γ_{γ_1} are diffraction phenomena. Only the $i\omega(t - x/\gamma_1)$ part of any exponent appears in the time factor; the h_1, h_2 terms modulate the amplitude. The result then is a superposition of pulses which appear to have travelled over the surface with the velocity of Rayleigh waves in medium I and proceed with undiminished amplitude.

For an initial P-pulse the horizontal displacements are asymmetrical and the vertical displacements symmetrical about time $t = x/\gamma_1$. The spread of a component is proportional to $(h_1/\widehat{\gamma_1\alpha_1} + h_2/\widehat{\gamma_1\beta_1})$. The greatest displacement is inversely proportional to the same factor so does not necessarily decrease with depth in the layer.

Not all terms give contributions but only those for which $\zeta = \omega/\gamma_1$ is a simple pole, that is, in which the expression F occurs as $1/F$ only. These seem to be characterized (for first- and higher-order terms) by a difference of sign attached to h and z where they occur in h_1 and h_2 in the exponent, that is, by a sort of asymmetry between the position of the source and of the observer.

(*f*) As $z \rightarrow 0$, that is, the point of reception approaches the surface, certain of the so-called 'blunt pulses' (those for which $h_1 = z$) increase in sharpness until for $z = 0$ they become part of a minimum-time path disturbance.

The possibility $h = 0$ is fully discussed later in connexion with the dispersive surface-wave motion, and the conclusion reached that the formulation of the problem is only valid for a source which is actually submerged.

(*g*) We have obtained a contribution corresponding to energy transfer from source to observer by each one of the minimum-time paths given by the ray theory. Further, those paths involving refraction at the critical angle (which on the simple ray theory are associated with zero energy) are found in fact to correspond to finite disturbances. It is evident from the way in which these results are arrived at mathematically that the corresponding results hold

(i) for any values of the velocities,

(ii) if the source and/or the point of reception is in the lower medium,

(iii) if the system is multilayered and the source and reception point located in any stratum.

We may therefore deduce the exponents which must occur in the ϕ and ψ series in any of these generalizations and hence the mathematical origin and, to some extent, the nature of the diffraction effects.

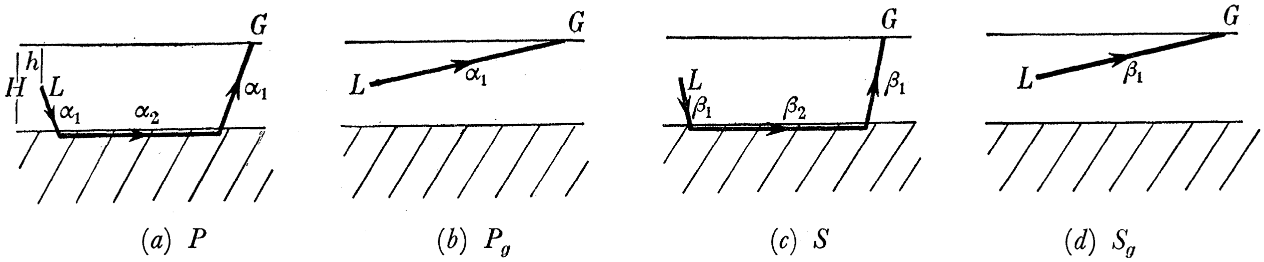

FIGURE 18. The P, P_g, S and S_g pulses of seismology.

We must now inquire into the relative importance of high-order contributions—first of the ray-path pulses. Although it is not immediately obvious because of the complicated nature of some of the coefficients, there is an amplitude reduction accompanying each successive reflexion or refraction just as the ray theory predicts. This may be readily verified whenever the path of travel involves a change of type, but when propagation is with constant velocity our assumption of x large means that angles of incidence and reflexion are all equal and almost 90°. Then our approximations conceal what is a very small but definite amplitude decrease accompanying each successive reflexion, but it is not difficult to locate the stage at which we suppress the relevant factor. In this connexion it should be remembered that a horizontal distance x which is sufficiently large for the determination of the once and twice reflected pulses by our approximate methods may not be 'large' with reference to a pulse ten times reflected. On the other hand, the amplitude of the latter is going to be small, and we interpret the failure of the approximation as indicating a 'blurring'. We may therefore expect a surface record, up to $t = x/\beta_2$, to show the following type of pattern when x is large:

The first arrival should be felt at $t = x/\alpha_2 + (2H-h)/\widehat{\alpha_1\alpha_2}$—the '*P*'- pulse of seismology—a sharp pulse which has travelled as in figure 18*a*, followed by a whole series of disturbances of the same type but diminishing amplitude and increasingly blurred.

At $t = x/\alpha_1 + h^2/2x\alpha_1$ the first of the Γ_{α_1}-pulses—the P_g of seismology—should be received. This is followed by a train of pulses which have undergone reflexion at the interface and surface; diminution in amplitude is at first less rapid for this group (for reasons given above) but becomes increasingly important and finally gives way to blurring. Sign differences between successive pulses should give an oscillatory character to the record.

Although we cannot picture the process in the case of the non-ray-path pulses from Γ_{β_2} and Γ_{β_1}, there appears with these to be the same steady decrease of amplitude with increasing order. In addition to the decrease which displays itself in the coefficient of each exponential, we saw from the study of ${}_p\phi^{(4)}_{\beta_2}$ and ${}_p\phi^{(4)}_{\beta_1}$ that there is a flattening and spreading of the blunt pulse as h_2 increases and a further time lag as h_1 increases. The Rayleigh pulses show this 'blunting' for increase of h_1 or h_2, but all are symmetrical or asymmetrical about time $t = x/\gamma_1$.

The diffraction phenomena are clearly going to modify considerably the passage of the other two trains of minimum-time path pulses. First would come those refracted along the interface with velocity β_2, the earliest arrival being the 'S' of seismology at time

$$t = x/\beta_2 + (2H-h)/\widehat{\beta_1\beta_2}$$

(figure 18c); next, those propagated entirely within the layer with velocity β_1, the first of these being S_g felt at time $t = x/\beta_1 + h_2/2x\beta_1$ (figure 18d).

Now if we specify the arrival of a diffraction pulse by the instant $\tau = 0$ about which it is roughly centred (see discussion of ${}_p\phi^{(4)}_{\beta_2}$), there will be such arrivals at

$$\begin{aligned}
&t = x/\beta_2; && p = (2H-h)/\widehat{\beta_2\alpha_1},\ (2H+h)/\widehat{\beta_2\alpha_1},\\
&t = x/\beta_2 + h/\widehat{\beta_1\beta_2}; && p = 2H/\widehat{\beta_2\alpha_1},\\
&t = x/\beta_2 + (H-h)/\widehat{\beta_1\beta_2}; && p = H/\widehat{\beta_2\alpha_1},\\
&t = x/\beta_2 + H/\widehat{\beta_1\beta_2}; && p = (H+h)/\widehat{\beta_2\alpha_1},\ (H-h)/\widehat{\beta_2\alpha_1},\\
&t = x/\beta_2 + (H+h)/\widehat{\beta_1\beta_2}; && p = H/\widehat{\beta_2\alpha_1}.
\end{aligned}$$

and the next arrival will be 'S' at

$$t = x/\beta_2 + (2H-h)/\beta_1\beta_2$$

(two values of p imply superposed pulses and p is defined as in the treatment of ${}_p\phi^{(4)}$).

If we suppose that by time $t = x/\beta_1$ any of the Γ_{β_2}-group of diffraction and ray-path pulses is severely blunted (and interference effects not yet important), then the instant $t = x/\beta_1$ should be marked by the arrival of the Γ_{β_1}-pulses, a superposition of diffraction pulses 'centred' at

$$\begin{aligned}
&t = x/\beta_1; && p = h/\widehat{\beta_1\alpha_1},\\
& && p = (2H-h)/\widehat{\beta_1\alpha_1},\ (2H+h)/\widehat{\beta_1\alpha_1},\\
&t = x/\beta_1 + h^2/2x\beta_1; && p = 2H/\widehat{\beta_1\alpha_1},
\end{aligned}$$

followed by the S_g of seismology at

$$t = x/\beta_1 + h^2/2x\beta_1.$$

If we ignore a time difference like $h^2/2x\beta_1$ in comparison with the spread of a diffraction pulse, then we have to superpose further diffraction pulses at

$$t \doteqdot x/\beta_1, \quad \text{with} \quad p = (H-h)/\widehat{\beta_1\alpha_1}, \quad H/\widehat{\beta_1\alpha_1}, \quad (H+h)/\widehat{\beta_1\alpha_1}, \quad 2H/\widehat{\beta_1\alpha_1}.$$

We may now be in a position to throw some light on the apparent S and S_g anomalies: (*a*) 'up to about 20° the S-residuals are spread over about 20 sec without any convincing concentration of frequency' (Jeffreys 1946, p. 61), (*b*) seismograms of near earthquakes sometimes show S_g as having arrived 1 to 2 sec before the time at which it would be expected after P_g (Jeffreys 1929, p. 98).

A difficulty now arises in the fact that having chosen to represent the generating disturbance by an instantaneous shock, we have derived infinite displacements at the time of arrival of certain of the minimum-time path pulses. We may overcome this failure to comply with physical conditions by slightly modifying the initial pulse to one of the form

$$\frac{1}{\pi}\tan^{-1}\frac{t}{s}=\frac{1}{\pi}\int_0^\infty e^{-s\omega}\sin\omega t\frac{d\omega}{\omega}$$

$$=\frac{1}{2\pi i}\int_{-\infty}^{\infty} e^{\mp s\omega+i\omega t}\frac{d\omega}{\omega}\quad(\mathscr{R}(\omega)\gtrless 0),\tag{10·2}$$

which for s small tends to

$$\left.\begin{array}{ll}+\frac{1}{2} & (t>0),\\ -\frac{1}{2} & (t<0),\end{array}\right\}\tag{10·3}$$

but varies continuously through the origin. About 70 % of the total change takes place between the values $\pm 2s$ of t (Lapwood 1949, p. 99). By choosing s sufficiently small we may therefore represent any very sudden but continuous displacement at the source.

Without further calculation it may be seen that all contributions will be of the form of so-called 'blunt pulses'; those which were originally ray-path pulses will be evaluated like ${}_p\phi^{(4)}_{\beta_2}$ and ${}_p\phi^{(4)}_{\beta_1}$, where the role of 'p' is fulfilled by 's', and those which were already of this second type will differ only in that p is replaced by $(p+s)$. Now we saw that $(1/p)$ was a measure of the greatest displacement and p a measure of the spread of such pulses. As $s\to 0$ the ray-path pulses become sharper, always finite but resembling more and more the form appropriate to the instantaneous shock (cf. the discussion of the diffraction effects when $h_1=z$ and $z\to 0$, p. 245). So long as the duration of the initial pulse is small compared with the 'p' of a resulting diffraction pulse then we may neglect the spread of any minimum-time path pulse compared with that of the diffraction pulse.

It seems reasonable then to suppose that at least some explanation of the S-anomaly is to be found in the arrival of exactly seven diffraction pulses before the true S-wave. If, as we suppose, the tail of the Γ_{α_1}-pulses is blurred and insignificant at time $t=x/\beta_2$, then the arrival of the first diffraction pulse might well be interpreted as the arrival of S. That there is no concentration of frequency of the residuals might simply be due to the 'personal fancy' of the individual observer; not convinced by the record of the first diffraction pulse he might interpret the second, third, ... (or the true S) as the onset. In support of this is the fact that, if h is just a fraction of H, the fourth, fifth, sixth and seventh pulses should certainly be sharper than the first to third, and the true S sharper than any. We have not needed to give any very exact estimate of when a diffraction pulse makes itself felt, i.e. what fraction of the maximum $\dot{U}$, $\dot{W}$ must be attained before it may be said to have 'arrived'. It is sufficient here to specify the arrival time by $\tau=0$, for we are obviously dealing with a quite considerable time difference between the first pulse and the true S. That interval is of the order of

$$(2H-h)/\widehat{\beta_1\beta_2}\ \text{sec.}$$

If we take for β_1 the value 3·4 km/sec, then $\widehat{\beta_1\beta_2}$ is 4·6 km/sec, and if the source is shallow we need a layer of only 35 km to give a residual of over 15 sec (this is just about the depth of combined sedimentary, granitic, and intermediate rock believed to overlie the ultra-basic in most continental areas).

Now consider the Γ_{β_1}-arrivals. We may note that the first listed diffraction pulse and the true S_g are 'zero-order' contributions and were therefore at the disposal of Lapwood for the elucidation of the S_g-anomaly; the diffraction pulse is his 'surface S-pulse'. If, as we suppose, the instants $t = x/\beta_1$ and $t = x/\beta_1 + h^2/2x\beta_1$ are inappreciably different, then a better estimate of the time of 'arrival' of the diffraction pulse is now essential. Lapwood (1949, p. 99) first reverted to the continuous pulse with time variation as

$$\frac{1}{\pi}\tan^{-1}\frac{t}{s} = \frac{1}{2\pi \mathrm{i}}\int_{-\infty}^{\infty} \mathrm{e}^{\mp s\omega + \mathrm{i}\omega t}\frac{\mathrm{d}\omega}{\omega} \quad (\mathscr{R}(\omega) \gtrless 0). \tag{10·4}$$

Combining the two contributions this gave

$$\dot{U} \doteqdot \frac{\sqrt{(2\beta_1)}\,\pi}{x^{\frac{3}{2}}}\left\{\frac{8\beta_1}{\widehat{\beta_1\alpha_1}}\frac{\sin(\frac{1}{2}\psi - \frac{1}{4}\pi)}{(s^2+\tau^2)^{\frac{1}{4}}} - \frac{h}{\beta_1}\frac{\sin(\frac{3}{2}\psi - \frac{1}{4}\pi)}{(s^2+\tau^2)^{\frac{3}{4}}} - \frac{4}{\widehat{\beta_1\alpha_1}}\frac{h^2}{x}\frac{\sin(\frac{3}{2}\psi + \frac{1}{4}\pi)}{(s^2+\tau^2)^{\frac{3}{4}}} - \frac{4\sin(\frac{1}{2}\psi' + \frac{1}{4}\pi)}{[(s+p)^2+\tau'^2]^{\frac{1}{4}}}\right\},$$

$$\dot{W} = \frac{2\sqrt{(2\beta_1)}}{x^{\frac{3}{2}}}\frac{\pi\beta_1}{\widehat{\beta_1\alpha_1}}\left\{\frac{8\beta_1}{\widehat{\beta_1\alpha_1}}\frac{\sin(\frac{1}{2}\psi + \frac{1}{4}\pi)}{(s^2+\tau^2)^{\frac{1}{4}}} - \frac{h}{\beta_1}\frac{\sin(\frac{3}{2}\psi + \frac{1}{4}\pi)}{(s^2+\tau^2)^{\frac{3}{4}}}\right.$$ (10·5)

$$\left. + \frac{4}{\widehat{\beta_1\alpha_1}}\frac{h^2}{x}\frac{\sin(\frac{3}{2}\psi - \frac{1}{4}\pi)}{(s^2+\tau^2)^{\frac{3}{4}}} + \frac{4\sin(\frac{1}{2}\psi' - \frac{1}{4}\pi)}{[(s+p)^2+\tau'^2]^{\frac{1}{4}}}\right\}, \tag{10·6}$$

where
$$\tau = t - x/\beta_1 - h^2/2x\beta_1, \quad \psi = \tan^{-1}(\tau/s), \tag{10·7}$$

$$\tau' = t - x/\beta_1, \quad p = h/\widehat{\beta_1\alpha_1} \quad \text{and} \quad \psi' = \tan^{-1}(\tau'/(s+p)), \tag{10·8}$$

the first three terms due to S_g and the last to the surface S-pulse. The functions

$$\cos^{\frac{1}{2}}\psi\,\sin(\tfrac{1}{2}\psi \pm \tfrac{1}{4}\pi)$$

attain 20 % of their maximum at approximately $\tau/s = -2$ and $+38$ and at $\tau/s = +2$ and -38 respectively. Lapwood therefore used the diffraction term of $\dot{W}$ to explain the apparent early arrival of the vertical component of S_g at $\tau/s = -38$ ($h/\widehat{\beta_1\alpha_1}$ now supposed $\ll s$), and the first term of $\dot{U}$ to explain the early arrival of the horizontal component at the same time.

This seems at once unnecessary and its validity doubtful. First, approximations known only to hold in the neighbourhood of $\tau = 0$ are used to support arguments at $\tau/s = -38$; indeed, it is shown by another method (Lapwood 1949, p. 89) that for the surface S-pulse figures 19 *a*, *b*, representing $\dot{U}$, $\dot{W}$, should be replaced by figures 20 *a*, *b*. These show the 20 % value as being attained at times given more nearly by ± 2; in a final note reference is made to this correction (Lapwood 1949, p. 99).

Next, Lapwood supposes that p for the surface S-pulse is small compared with s and accounts for the anomaly in the horizontal by the spread of S_g itself due to the finite duration of the initial shock. As the anomaly is a 'relative' one (the early arrival of S_g after P_g), and it can be shown that P_g will show a comparable spread to the fore (and in both $\dot{U}$ *and* $\dot{W}$), this argument would seem to lose its weight. Rather, we would seek an explanation which is quite independent of the finite duration of the initial shock.

Now the spread of S_g is measured by s and the spread of the surface S-pulse by $h/\widehat{\beta_1\alpha_1}+s$ (the respective p values). If 20 % of $\dot{U}_{\max.}$, $\dot{W}_{\max.}$ is supposed still to signify an arrival then S arrives of the order of $2h/\widehat{\beta_1\alpha_1}$ sec in advance of S_g and might be interpreted as S_g so long as S_g is at this stage still inappreciable (submergence to a depth of 5 km would give an error of about 2 sec). Otherwise we must turn to the diffraction phenomena attributable to stratification.

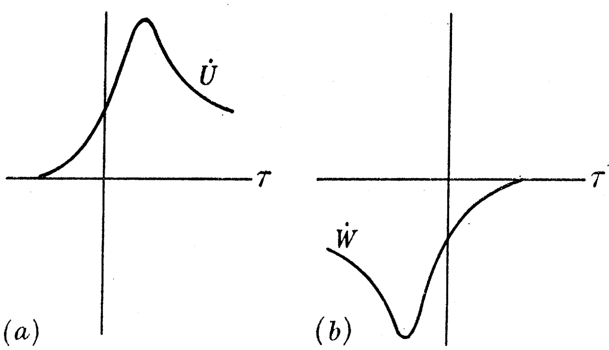

FIGURE 19. Approximation to variation of (*a*) $\dot{U}$, (*b*) $\dot{W}$ in 'surface S-pulse' (related to figures 11*a* and *b*).

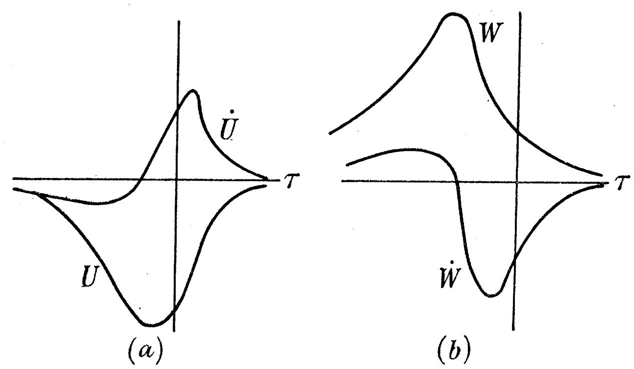

FIGURE 20. Better approximation to the variation of (*a*) $\dot{U}$, (*b*) $\dot{W}$ in the 'surface S-pulse'.

It appears true of any component that improving the approximations to $\dot{U}_{\beta_1}$, $\dot{W}_{\beta_1}$ (see appendix 1) gives a steep descent from $\dot{U}_{\max.}$, $\dot{W}_{\max.}$ in the directions of τ both increasing and decreasing; consequently, $\tau/(p+s) = -2$ may be taken as roughly the stage at which any diffraction pulse attains 20 % of its maximum rates of displacement. With the single surface layer we found superposed about the instant $t = x/\beta_1$ blunt pulses with p-values $(H-h)/\widehat{\beta_1\alpha_1}$, $H/\widehat{\beta_1\alpha_1}$, $(H+h)/\widehat{\beta_1\alpha_1}$, $(2H-h)/\widehat{\beta_1\alpha_1}$, $(2H+h)/\widehat{\beta_1\alpha_1}$. Neglecting h compared with H and using the 20 % rule, the first of these could account for a time error of the order of $2H/\widehat{\beta_1\alpha_1}$. This can only be regarded as suggesting an order of magnitude (15 sec for a layer of 30 km); it may be that some value nearer 100 % of $\dot{U}_{\max.}$, $\dot{W}_{\max.}$ is attained before an 'arrival' is recorded, if this simple model is even applicable. Moreover, the zero- and first-order terms provide only a very incomplete sample of the diffraction effects; an investigation of higher-order contributions would clearly produce further pulses for which $\tau \doteqdot x/\beta_1$ and with $p = (3H-h)/\widehat{\beta_1\alpha_1}, \ldots$, their 'spread' increasing and magnitude decreasing with increasing p. The apparent early arrival of S_g may well be attributable to confusion, not with any single diffraction pulse, but with some more important interference effect. In part II we shall study in detail the interference pattern at great range.

11. Steepest descent method: least distance criteria for individual pulses

So far, we have not used the methods of steepest descent or stationary phase for the approximate evaluation of integrals.

Indeed, the original ζ-integrals, of the type

$$\phi = \int_0^\infty \frac{[\overline{S}+\overline{T}\exp\{-2H\lambda_{\beta_1}\}+(\overline{V}+\overline{Y})\exp\{-H(\lambda_{\alpha_1}+\lambda_{\beta_1})\}+\ldots]}{[S'+T'\exp\{-2H\lambda_{\beta_1}\}+\ldots]}\exp\{-h\lambda_{\alpha_1}\}\cos\zeta x\,\mathrm{d}\zeta$$

are not of a suitable form because of the occurrence of the exponentials within the first factor. When, however, we have applied the Bromwich expansion principle each resulting term of the series for ${}_p\phi$, ${}_p\psi$, ${}_s\phi$, ${}_s\psi$ is of the type

$$\begin{aligned}\phi &= 2\int_0^\infty \bar{\chi}(\zeta)\exp\{\mathrm{i}\omega t-h_1\lambda_{\alpha_1}-h_2\lambda_{\beta_1}\}\begin{matrix}\cos\zeta x\\ \sin\zeta x\end{matrix}\,\mathrm{d}\zeta \quad \left(\bar{\chi}(\zeta)\begin{cases}\text{even}\\ \text{odd}\end{cases}\text{in }\zeta\right),\\ &= \int_{-\infty}^\infty \chi(\zeta)\exp\{\mathrm{i}\omega t-\mathrm{i}\zeta x-h_1\lambda_{\alpha_1}-h_2\lambda_{\beta_1}\}\,\mathrm{d}\zeta, \quad \text{where} \quad \chi(\zeta)=\begin{cases}\bar{\chi}(\zeta)\\ -\mathrm{i}\bar{\chi}(\zeta)\end{cases}\\ &= \int_{-\infty}^\infty \chi(\zeta)\,\mathrm{e}^{f(\zeta)}\,\mathrm{d}\zeta, \qquad (11\cdot1)\end{aligned}$$

where $\chi(\zeta)$ varies slowly relative to the exponential factor so long as $|\omega|$ is not too small, and the methods of stationary phase or steepest descents may be applied.

Nakano (1925) used both methods in the simpler problem of a line source in a uniform medium, and although his treatment did not admit the simple physical interpretation derived by Lapwood using the Sommerfeld contour, more information was obtained about the disturbance at short range. Primarily, we are interested in great range, but it is of interest to know at what distances we may expect individual pulses to be first felt. Nakano's methods produced certain least-distance criteria for the Rayleigh pulse, 'surface S-pulse', and 'surface P-pulse'. Lapwood (1949) observed that these were the same conditions as determined the stages at which the steepest-descent curve, when confined to the top leaf of the Riemann surface, has to be distorted to avoid a pole or branch line, i.e. the stages at which it acquires, one by one, the essential features of the Sommerfeld contour. The curve of stationary phase, even with slight modifications, cannot be confined to the top leaf, and although the same conclusions were reached, the discussion was more difficult.

Anticipating a similar situation we shall approximate to the integral (11·1) by 'steepest descents'. If the original contour, the real axis, can be distorted into the line of steepest descent by arcs which give negligible contribution, then the main contribution to the integral comes from the region of the saddle-point on the steepest-descent curve and is approximately

$$\surd(2\pi/f''(\zeta_0))\,\chi(\zeta_0)\exp\{f(\zeta_0)+\mathrm{i}\alpha\}, \qquad (11\cdot2)$$

where ζ_0 is the saddle-point and α the inclination of the curve there to the real axis; the curve is defined by $\mathscr{I}[f(\zeta)]=$ constant, described so that $\mathscr{R}[f(\zeta)]$ decreases along it.

For the purpose of this discussion we shall consider $\mathscr{R}(\omega)>0$, that is,

$$\omega = s-\mathrm{i}c \quad (s>0,\ c>0),$$

and suppose that the integrand

$$\phi \equiv \int_{-\infty}^\infty \chi(\zeta)\,\mathrm{e}^{f(\zeta)}\,\mathrm{d}\zeta,$$

where
$$f(\zeta) = \mathrm{i}\omega t-\mathrm{i}\zeta x-h_1(\zeta^2-\kappa_{\alpha_1}^2)^{\frac{1}{2}}-h_2(\zeta^2-\kappa_{\beta_1}^2)^{\frac{1}{2}},$$

has branch lines $\mathscr{R}(\lambda_{\alpha_2})=0$, $\mathscr{R}(\lambda_{\alpha_1})=0$, $\mathscr{R}(\lambda_{\beta_2})=0$, $\mathscr{R}(\lambda_{\beta_1})=0$ and a pole κ_{γ_1}. Put

$$\zeta = \kappa_{\alpha_1}\theta,$$

then, with the velocity ratios used throughout, namely,

$$\kappa_{\alpha_2}:\kappa_{\alpha_1}:\kappa_{\beta_2}:\kappa_{\beta_1} = \tfrac{3}{4}:1:\tfrac{3}{4}\surd3:\surd3,$$

the branch points are, respectively,

$$\theta_{\alpha_2} = \tfrac{3}{4}, \quad \theta_{\alpha_1} = 1, \quad \theta_{\beta_2} = \tfrac{3}{4}\surd 3, \quad \theta_{\beta_1} = \surd 3, \tag{11·3}$$

and the pole κ_{γ_1} is

$$\theta_{\gamma_1} = \surd 3/0{\cdot}9194\ldots. \tag{11·4}$$

The saddle-point is the point θ_0, where

$$\frac{\mathrm{d}f}{\mathrm{d}\zeta} \equiv -\mathrm{i}x - \frac{\mathrm{i}h_1\theta}{\surd(1-\theta^2)} - \frac{\mathrm{i}h_2\theta}{\surd(\alpha_1^2/\beta_1^2-\theta^2)} = 0,$$

giving

$$x = h_1\theta_0/(1-\theta_0^2)^{\frac{1}{2}} + h_2\theta_0/(3-\theta_0^2)^{\frac{1}{2}}. \tag{11·5}$$

This has only one real root, which increases with x and lies between $\theta = 0$ and $\theta = 1$. The saddle-point is therefore on the line of branch points.

The line of steepest descent is determined by the relation

$$\mathscr{I}[\mathrm{i}\zeta x + \mathrm{i}h_1(\kappa_{\alpha_1}^2-\zeta^2)^{\frac{1}{2}} + \mathrm{i}h_2(\kappa_{\beta_1}^2-\zeta^2)^{\frac{1}{2}}] = \text{constant (value at saddle-point)}$$
$$= (s/\alpha_1)\,[x\theta_0 + h_1(1-\theta_0^2)^{\frac{1}{2}} + h_2(3-\theta_0^2)^{\frac{1}{2}}]. \tag{11·6}$$

If we write

$$\zeta = \xi + \mathrm{i}\eta,$$

then points on the curve for which $|\zeta|$ is large are given by

$$\xi x \pm (h_1+h_2)\,\eta = 0 \quad (\mathscr{R}(\zeta) \gtrless 0), \tag{11·7}$$

portions of two straight lines in the fourth and third quadrants respectively.

Close to the saddle-point we may write

$$\zeta = \kappa_0 + \xi' + \mathrm{i}\eta' \quad (\kappa_0 = \kappa_{\alpha_1}\theta_0), \tag{11·8}$$

and obtain an approximation to the curve for ξ', η' small by considering one or more terms of the Taylor expansion of $f(\zeta)$. Since κ_0 is a stationary point and by definition is on the curve of steepest descent, that curve is given by

$$[(\xi'+\mathrm{i}\eta')^2 f''(\kappa_0)/2! + (\xi'+\mathrm{i}\eta')^3 f'''(\kappa_0)/3! + \ldots] = \text{a real and negative quantity.}$$

As a first approximation we have, since $f''(\kappa_0)$ is positive imaginary,

$$(\xi'+\mathrm{i}\eta')^2\,\mathrm{i}/(s-\mathrm{i}c) = \text{a real and negative quantity}$$

or

$$\begin{cases} s\xi'^2 - 2c\xi'\eta' - s\eta'^2 = 0, & (11{\cdot}9) \\ c\xi'^2 + 2s\xi'\eta' - c\eta'^2 > 0. & (11{\cdot}10) \end{cases}$$

Equation (11·9) breaks up into two straight lines

$$[\eta' - \xi'\{-c+\surd(s^2+c^2)\}/s]\,[\eta' + \xi'\{c+\surd(s^2+c^2)\}/s] = 0, \tag{11·11}$$

of which, by (11·10), the first only is relevant and represents the tangent to the line of steepest descent on the top leaf of the Riemann surface.

The next approximation is

$$(\xi'+\mathrm{i}\eta')^2 f''(\kappa_0)/2! + (\xi'+\mathrm{i}\eta')^3 f'''(\kappa_0)/3! = (\text{real and negative})$$

or

$$3[s(\xi'^2-\eta'^2) - 2c\xi'\eta']\,\omega f'''(\kappa_0)/f''(\kappa_0) - [-(3\xi'^2\eta'-\eta'^3)\,2sc + (\xi'^3-3\xi'\eta'^2)\,(s^2-c^2)]\,\alpha_1/(s^2+c^2)$$
$$= (\text{real and negative}),$$

and putting
$$\zeta' = \frac{3f'''(\kappa_0)\,\omega}{f''(\kappa_0)}\,\zeta,$$
the condition that the imaginary part of $f(\zeta)$ vanishes may be written
$$[\bar{\eta}+\bar{\xi}(s/\alpha_1-\bar{\xi})\,\{\surd(s^2+c^2)+c\}/s^2]\,[\bar{\eta}+\bar{\xi}(s/\alpha_1-\bar{\xi})\,\{c-\surd(s^2+c^2)\}/s^2] = 0, \qquad (11\cdot12)$$
where we have partially substituted for $\bar{\eta}$ from the first approximation and have added fourth-order terms, supposing them small. We deduce that in the neighbourhood of the saddle-point the curve is part of a parabola which cuts the $\bar{\xi}$-axis again at s/α_1 and whose equation is
$$\bar{\eta} = \bar{\xi}(s/\alpha_1-\bar{\xi})\,\{\surd(s^2+c^2)-c\}/s^2, \qquad (11\cdot13)$$
but we find that for no value of x is this ever more than a local approximation. We may suppose, however, that the curve is of roughly parabolic shape with modifications in the neighbourhood of the apex.

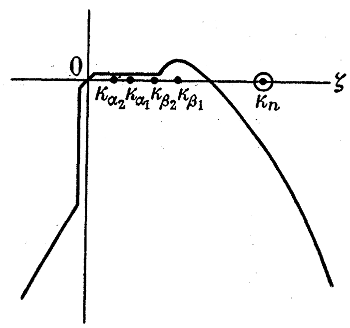

FIGURE 21. Modified steepest-descent curve (ω real).

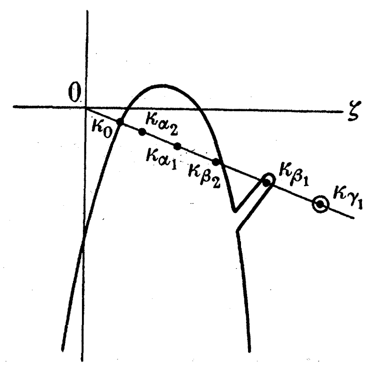

FIGURE 22. Modified steepest-descent curve (ω complex).

For real ω and $\zeta = \kappa_{\alpha_1}(u+iv)$, the curve is given exactly by
$$\mathscr{I}[ix(u+iv)+ih_1\{1-(u+iv)^2\}^{\frac{1}{2}}+ih_2\{3-(u+iv)^2\}^{\frac{1}{2}}] = \text{constant},$$
that is,
$$xu+h_1\surd\{\tfrac{1}{2}(1-u^2+v^2)+\tfrac{1}{2}[(1-u^2+v^2)^2+4u^2v^2]^{\frac{1}{2}}\}$$
$$+h_2\surd\{\tfrac{1}{2}(3-u^2+v^2)+\tfrac{1}{2}[(3-u^2+v^2)^2+4u^2v^2]^{\frac{1}{2}}\} = \text{constant} \qquad (11\cdot14)$$
where the positive roots are taken. Since
$$\mathscr{R}(\lambda_{\alpha_1}) = (s/\alpha_1)\,uv/\surd\{\tfrac{1}{2}(1-u^2+v^2)+\tfrac{1}{2}[(1-u^2+v^2)^2+4u^2v^2]^{\frac{1}{2}}\}, \qquad (11\cdot15)$$
and
$$\mathscr{R}(\lambda_{\beta_1}) = (s/\alpha_1)\,uv/\surd\{\tfrac{1}{2}(3-u^2+v^2)+\tfrac{1}{2}[(3-u^2+v^2)^2+4u^2v^2]^{\frac{1}{2}}\}, \qquad (11\cdot16)$$
only those portions of the curve for which $uv>0$ lie on the top leaf of the Riemann surface and the curve must be distorted to avoid regions $uv<0$, equivalent we see (figure 21) to avoiding crossing one or more of the superposed branch lines.

We shall see that for our purpose it is not necessary to know the detailed form of the steepest-descent curve; it suffices to know the number and location of the points in which it intersects the line of branch points. Assuming a curve of parabolic type we shall examine in detail the variation of $\mathscr{I}[f(\zeta)]$ along this line, deduce the essential modifications to this supposed form of curve and draw what conclusions we may from its relation to the Sommerfeld contour. If at a certain distance x the curve should take the form of figure 22 (where the narrow loop is a necessary detour to avoid crossing the branch line $\mathscr{R}(\lambda_{\beta_1}) = 0$ and the small circle surrounds the pole), we should conclude that, at this range, both the Γ_{γ_1} and the Γ_{β_1} pulses are significant relative to the saddle-point contribution.

Writing $\zeta = \kappa_{\alpha_1}\theta$ we examine the variation of $\mathscr{I}[-f(\zeta)]$ for real θ:

(1) $0 \leqslant \theta \leqslant 1$, $\omega = s - ic$ $(s > 0, c > 0)$.

We have

$$\mathscr{I}[-f(\zeta)] = (s/\alpha_1)\,[x\theta + h_1\sqrt{(1-\theta^2)} + h_2\sqrt{(\alpha_1^2/\beta_1^2 - \theta^2)}]. \tag{11·17}$$

There is a maximum at θ_0, the saddle-point, given by

$$x - h_1\theta_0/\sqrt{(1-\theta_0^2)} - h_2\theta_0/\sqrt{(3-\theta_0^2)} = 0. \tag{11·18}$$

The left-hand side of (11·18) is the derivative $\dfrac{d}{d\zeta}[\mathscr{I}(-f)]$, and hence the branch-point κ_{α_2} $(\theta_{\alpha_2} = \alpha_1/\alpha_2 = \frac{3}{4})$ lies to the left or to the right of the saddle-point according as

$$x \gtrless h_1\theta_{\alpha_2}/\sqrt{(1-\theta_{\alpha_2}^2)} + h_2\theta_{\alpha_2}/\sqrt{(\alpha_1^2/\beta_1^2 - \theta_{\alpha_2}^2)}, \tag{11·19}$$

that is, distortion will be necessary and the Γ_{α_2} pulse will become significant as soon as

$$x > h_1/\sqrt{(\alpha_2^2/\alpha_1^2 - 1)} + h_2/\sqrt{(\alpha_2^2/\beta_1^2 - 1)}$$
$$= 3h_1/\sqrt{7} + 3h_2/\sqrt{39}. \tag{11·20}$$

(2) $1 \leqslant \theta \leqslant \alpha_1/\beta_1$ $(= \sqrt{3})$. The steepest-descent curve recuts the axis between 1 and $\sqrt{3}$ if $\mathscr{I}(-f)$ equals or exceeds the saddle-point value there.

We have that

$$\mathscr{I}(-f) = (s/\alpha_1)\,[x\theta + h_2\sqrt{(3-\theta^2)}] - (c/\alpha_1)\,h_1\sqrt{(\theta^2-1)}, \tag{11·21}$$

so that if it attains the saddle-point value for any ω it will do so for $c = 0$. In particular, then, we study the variation of

$$(s/\alpha_1)\,[x\theta + h_2\sqrt{(3-\theta^2)}] \quad (1 \leqslant \theta \leqslant \sqrt{3}). \tag{11·22}$$

The expression has a single maximum at

$$\theta = \sqrt{3}\,x/(x^2+h_2^2)^{\frac{1}{2}}, \tag{11·23}$$

which lies within the range if

$$\sqrt{3}\,x/(x^2+h_2^2)^{\frac{1}{2}} > 1, \quad \text{that is,} \quad x > h_2/\sqrt{2}. \tag{11·24}$$

Consider in turn the cases

(*a*) $x < h_2/\sqrt{2}$. The expression (11·22) decreases from the value

$$(s/\alpha_1)\,(x + \sqrt{2}\,h_2) \quad \text{at} \quad \theta = 1$$

to

$$(s/\alpha_1)\,\sqrt{3}\,x \quad \text{at} \quad \theta = \sqrt{3},$$

and we conclude that there is no intersection in the range $\theta = 1$ to $\sqrt{3}$. Thus for $x < h_2/\sqrt{2}$ the Γ_{β_1}, Γ_{β_2} pulses are not significant.

Extending the argument to $\theta > \sqrt{3}$ we see that a second intersection must occur when

$$x\theta = x\theta_0 + h_1(1-\theta_0^2)^{\frac{1}{2}} + h_2(3-\theta_0^2)^{\frac{1}{2}}. \tag{11·25}$$

Therefore the Rayleigh pulse (from Γ_{γ_1}) will be felt when

$$x\alpha_1/\gamma_1\,(= \sqrt{3}\,x/0{\cdot}9194\ldots) > x\theta_0 + h_1(1-\theta_0^2)^{\frac{1}{2}} + h_2(3-\theta_0^2)^{\frac{1}{2}}. \tag{11·26}$$

Although θ_0 is a function of x, it is easily shown that this relation is equivalent to a simple one of the type $x > x_0$.

(*b*) $x > h_2/\sqrt{2}$. There are three possibilities.

Case I. The value at the maximum is less than the saddle-point value, that is,

$$\sqrt{3}\,(x^2+h_2^2)^{\frac{1}{2}} < x\theta_0 + h_1(1-\theta_0^2)^{\frac{1}{2}} + h_2(3-\theta_0^2)^{\frac{1}{2}}, \tag{11·27}$$

and there is no intersection in $1 < \theta < \sqrt{3}$. The situation is essentially as in (*a*).

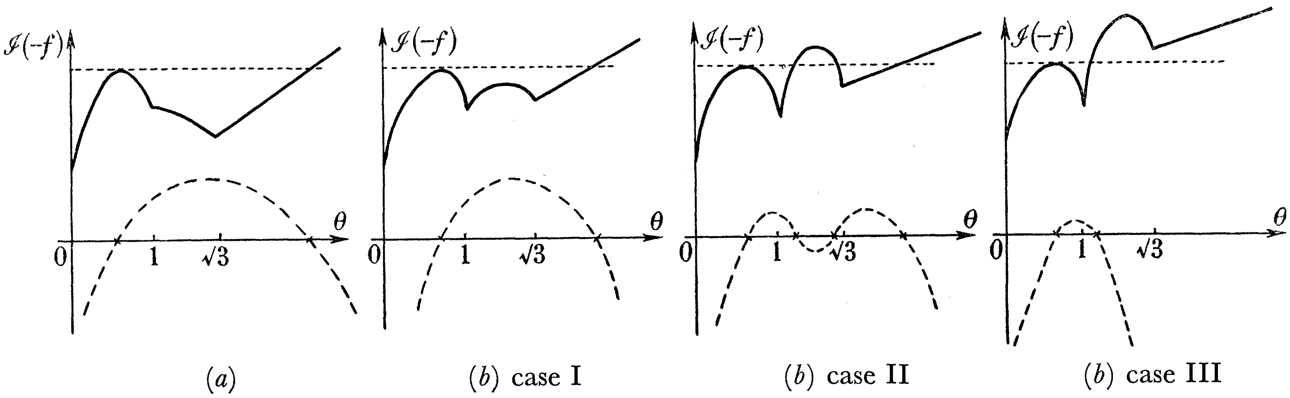

FIGURE 23. The variation of $\mathscr{I}(-f)$ with θ (—) and the steepest-descent curve (---). (*a*) $x < h_2/\sqrt{2}$, (*b*) $x > h_2/\sqrt{2}$.

Case II. The saddle-point value is less than the value at the maximum but greater than the value at $\sqrt{3}$, that is,

$$\sqrt{3}\,(x^2+h_2^2)^{\frac{1}{2}} > x\theta_0 + h_1\,(1-\theta_0^2)^{\frac{1}{2}} + h_2\,(3-\theta_0^2)^{\frac{1}{2}} > \sqrt{3}\,x, \tag{11·28}$$

so that the steepest-descent curve makes three other intersections with the line of branch points. The Γ_{β_1}-pulse does not appear; the Γ_{β_2}-pulse is significant if

$$x\alpha_1/\beta_2 + h_2\,(3-\alpha_1^2/\beta_2^2)^{\frac{1}{2}} > x\theta_0 + h_1\,(1-\theta_0^2)^{\frac{1}{2}} + h_2\,(3-\theta_0^2)^{\frac{1}{2}}, \tag{11·29}$$

into which relation may be substituted $\alpha_1/\beta_2 = \frac{3}{4}\sqrt{3}$. The Γ_{γ_1}-pulse is felt according to (11·26).

Case III. The saddle-point value is less than the value at the maximum and less than the value at $\sqrt{3}$. Since for $c = 0$, θ real and $> \sqrt{3}$ we have

$$\mathscr{I}(-f) = (s/\alpha_1)\,x\theta,$$

it follows that $\mathscr{I}(-f)$ continually increases from $\theta = \sqrt{3}$, and we see that there can be only one intersection in $(1, \sqrt{3})$ and none for $\theta > \sqrt{3}$. Thus the Γ_{γ_1}- and Γ_{β_1}-pulses are necessarily significant and the Γ_{β_2}-pulses also if

$$x\alpha_1/\beta_2 + h_2\,(3-\alpha_1^2/\beta_2^2)^{\frac{1}{2}} > x\theta_0 + h_1\,(1-\theta_0^2)^{\frac{1}{2}} + h_2\,(3-\theta_0^2)^{\frac{1}{2}}. \tag{11·30}$$

It is now apparent that the order of appearance of the Γ_{α_2}-, Γ_{β_2}-, Γ_{β_1}-, Γ_{γ_1}-pulses is not fixed but depends on the values of h_1, h_2, α_2, α_1, β_2, β_1. We must first check that the three cases treated geometrically above do, in fact, arise in the order I to III as x is continuously increased.

At any given θ the function $\mathscr{I}(-f)$ increases linearly with x at a rate proportional to θ, so that although the saddle-point varies, the value of $\mathscr{I}(-f)$ there is certainly increasing at a lesser rate (with respect to x) than at a point for which $\theta > 1$; and for $\theta > 1$, the greater θ then the greater the rate. It is now readily seen that cases I to III must arise in that order; likewise that any one of the above conditions for the appearance of a particular pulse must be equivalent to a simple minimum distance criterion, $x > x_0$.

We are now in a position to state that the Γ_{β_2}- or the Γ_{β_1}-pulse is felt first according as the value of $\mathscr{I}(-f)$ at κ_{β_2} is greater than or less than its value at the saddle-point when the value at κ_{β_1} first attains the value at the saddle-point (see figure 23), that is, according as

$$x\alpha_1/\beta_2 + h_2(\alpha_1^2/\beta_1^2 - \alpha_1^2/\beta_2^2)^{\frac{1}{2}} \gtrless x\theta_0 + h_1(1-\theta_0^2)^{\frac{1}{2}} + h_2(\alpha_1^2/\beta_1^2 - \theta_0^2)^{\frac{1}{2}}, \tag{11·31}$$

where

$$x = h_1\theta_0/(1-\theta_0^2)^{\frac{1}{2}} + h_2\theta_0/(\alpha_1^2/\beta_1^2 - \theta_0^2)^{\frac{1}{2}} \tag{11·32}$$

and

$$x\alpha_1/\beta_2 = x\theta_0 + h_1(1-\theta_0^2)^{\frac{1}{2}} + h_2(\alpha_1^2/\beta_1^2 - \theta_0^2)^{\frac{1}{2}}. \tag{11·33}$$

These relations are reducible to a condition in h_1, h_2 and the wave-velocities of the media.

Summarizing our results we have that

(a) the Γ_{α_2}-pulse appears when

$$x > h_1\alpha_1/\alpha_2(1-\alpha_1^2/\alpha_2^2)^{\frac{1}{2}} + h_2\alpha_1/\alpha_2(\alpha_1^2/\beta_1^2 - \alpha_1^2/\alpha_2^2)^{\frac{1}{2}},$$

(b) the Γ_{β_2}-pulse when

$$x\alpha_1/\beta_2 + h_2(\alpha_1^2/\beta_1^2 - \alpha_1^2/\beta_2^2)^{\frac{1}{2}} > x\theta_0 + h_1(1-\theta_0^2)^{\frac{1}{2}} + h_2(\alpha_1^2/\beta_1^2 - \theta_0^2)^{\frac{1}{2}},$$

which certainly implies $x > h_2/\sqrt{2}$,

(c) the Γ_{β_1}-pulse when

$$x\alpha_1/\beta_1 > x\theta_0 + h_1(1-\theta_0^2)^{\frac{1}{2}} + h_2(\alpha_1^2/\beta_1^2 - \theta_0^2)^{\frac{1}{2}},$$

which again implies $x > h_2/\sqrt{2}$,

(d) the Γ_{γ_1}-pulse when

$$x\alpha_1/\gamma_1 > x\theta_0 + h_1(1-\theta_0^2)^{\frac{1}{2}} + h_2(\alpha_1^2/\beta_1^2 - \theta_0^2),$$

(e) if substituting $\theta_0 = \alpha_1/\alpha_2$ $(= \theta_{\alpha_2})$ none of the relations (b), (c), (d) is satisfied, we conclude that the Γ_{α_2}-pulse (corresponding to partial travel along the interface with velocity α_2) is felt first,

(f) the Γ_{β_1}-pulse precedes the Γ_{β_2}-pulse if

$$x\alpha_1/\beta_2 + h_2(\alpha_1^2/\beta_1^2 - \alpha_1^2/\beta_2^2) < x\theta_0 + h_1(1-\theta_0^2)^{\frac{1}{2}} + h_2(\alpha_1^2/\beta_1^2 - \theta_0^2)^{\frac{1}{2}},$$

when

$$x\alpha_1/\beta_1 = x\theta_0 + h_1(1-\theta_0^2)^{\frac{1}{2}} + h_2(\alpha_1^2/\beta_1^2 - \theta_0^2)^{\frac{1}{2}}$$

and θ_0 is defined as throughout. Geometrically, we see that if this is so, the Γ_{γ_1}-, Γ_{β_1}-, Γ_{β_2}-pulses appear in that order; but if the Γ_{β_2} precedes the Γ_{β_1} the order is not fixed.

It remains now to examine the saddle-point contribution as given by (11·2). It contains the factor

$$\exp\{i\omega\{t - x\theta_0/\alpha_1 - h_1(1-\theta_0^2)^{\frac{1}{2}}/\alpha_1 - h_2(\alpha_1^2/\beta_1^2 - \theta_0^2)^{\frac{1}{2}}/\alpha_1\}\}, \tag{11·34}$$

where θ_0 is given by

$$x = h_1\theta_0/(1-\theta_0^2)^{\frac{1}{2}} + h_2\theta_0/(\alpha_1^2/\beta_1^2 - \theta_0^2)^{\frac{1}{2}}. \tag{11·35}$$

Substituting $\theta = \cos i$, it is readily shown that (11·35) expresses the condition and (11·34) contains the appropriate time-factor for travel by a minimum-time path, partly as a compressional and partly as a distortional wave in the layer. The lengths h_1 and h_2 are necessarily such that a complete path between source and observer is derived and i is the angle of incidence (as a compressional wave) at the surface or interface. We obtain, in fact, the equivalent of the 'Γ_{α_1}'-contribution, and may conclude, therefore, that these so-called 'Γ_{α_1}'-disturbances should be felt however close to the source.

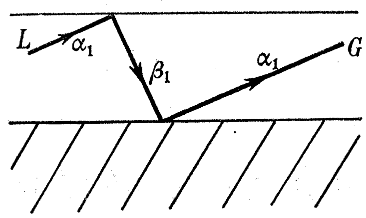

FIGURE 24. Saddle-point contribution ($h_1 = H+h-z$, $h_2 = H$).

When $h_2 = 0$ the above arguments remain valid, for the steepest-descent curve still contracts about $\theta = 1$ and the fact that $\mathscr{I}(-f)$ (ω real) now increases linearly from $\theta = 1$ (not $\theta = \sqrt{3}$, as before) simply cuts out the second possibility of two intersections in $1 < \theta < \sqrt{3}$; the Γ_{β_1}-pulses must therefore always precede the Γ_{β_2}. This case was treated in part by Lapwood in the problem of the uniform medium, but there were naturally no λ_{α_2}- and λ_{β_2}-branch lines to consider.

When $h_1 = 0$ the arguments must be slightly altered as the steepest-descent curve now contracts about $\theta = \sqrt{3}$ (i.e. κ_{β_1}); the Γ_{α_2}-, Γ_{α_1}-, Γ_{β_2}-pulses become significant in that order and the Γ_{γ_1}-pulse at an earlier, later, or intermediate stage depending on the velocity ratios only. In this case the Γ_{β_1}-pulses arise from the saddle-point contribution and should therefore be felt at all distances.

Our results have been obtained for a harmonic vibration of period $2\pi/\omega$, where, to ensure the relatively slow variation of the factor $\chi(\zeta)$ in the original integral, $|\omega|$ was assumed not too small. On generalizing to an initial pulse we may expect the same characteristics to be observed provided contributions from small $|\omega|$ are unimportant. The term 'period' is used rather loosely to include complex ω. We have also referred to contributions as Γ_{β_1}-, Γ_{β_2}-, ... 'pulses' throughout the main arguments, that is, before actually generalizing to an initial pulse.

12. THE CONTRIBUTIONS FROM THE STONELEY POLES

We have reserved until now the discussion of the equation

$$S(\zeta) \equiv \begin{vmatrix} -\zeta & \lambda_{\beta_1} & -\zeta & \lambda_{\beta_2} \\ \lambda_{\alpha_1} & -\zeta & -\lambda_{\alpha_2} & -\zeta \\ -2\zeta\lambda_{\alpha_1} & (2\zeta^2-\kappa_{\beta_1^2}) & 2\zeta\lambda_{\alpha_2}\mu_2/\mu_1 & (2\zeta^2-\kappa_{\beta_2^2})\,\mu_2/\mu_1 \\ (2\zeta^2-\kappa_{\beta_1^2}) & -2\zeta\lambda_{\beta_1} & (2\zeta^2-\kappa_{\beta_2^2})\,\mu_2/\mu_1 & 2\zeta\lambda_{\beta_2}\mu_2/\mu_1 \end{vmatrix} = 0, \qquad (12\cdot1)$$

and the possibility of contributions from the Stoneley poles. The equation (12·1) was shown by Stoneley (1924) to determine the possible systems of free waves which may be propagated over the interface between two distinct media of infinite depth. By distorting the ω-contour into an infinite semicircle, so making the 'wave-length' infinitesimally small, we have in effect reproduced these conditions in our own problem.

Scholte (1942), considering real ω, treated the algebra of (12·1) fairly exhaustively and showed that the equation has 2, 1 or 0 real roots according to the values of the elastic constants and densities. He examined the equation in its original form; only real roots in ζ correspond to true Stoneley waves, that is, unattenuated motion over the interface.

In order to investigate the possibility of complex roots the equation (12·1) was rationalized yielding a sixteenth-order equation in Y, where

$$Y = (\zeta/\kappa_{\alpha_2})^2. \tag{12·2}$$

To simplify the arithmetic in the early stages, the assumed earth model was replaced by one in which

$$\alpha_2:\alpha_1:\beta_2:\beta_1 = \sqrt{6}:\sqrt{3}:\sqrt{2}:1. \tag{12·3}$$

A programme was prepared for the solution of this sixteenth-order equation on an electronic calculator.* The method was an adaptation of Newton's method and gave six real roots and five pairs of conjugate complex roots. We are interested only in those solutions which lie on the top leaf of the Riemann surface. As predicted by Scholte, for our choice of elastic constants the equation (12·1) has no real roots; in addition, none of the complex roots is relevant. It would seem then that in this particular case the Stoneley poles make no contribution to the disturbance.

The chance of error in the rationalization and subsequent solution of (12·1) is, however, appreciable, and the possibility of handling such an equation in its original form is being investigated. It may then be possible to study the existence of roots for a whole range of elastic constants, a necessary step before the role of the Stoneley poles can be properly understood.

13. The disturbance at great range

We have found in the Bromwich expansion method a convenient means of describing the disturbance due to the line source in terms of a succession of pulses whose existence may be deduced from a simple ray theory and which correspond to travel by minimum-time paths within the surface layer and along the interface in the lower medium. As distinct from the analogous Love-wave problem (Jeffreys 1931), there is additional motion not explicable in terms of ray paths and corresponding generally to blunt movements.

Although the accuracy of our approximations has depended for the most part on a large horizontal range x, it was remarked that each single pulse might best be distinguishable as such at the minimum distance for its appearance. At moderate range interference between pulses must begin and at very great range we may expect the pattern of pulses to be partly or even wholly lost and the disturbance governed by these interference effects. Now from considerations of energy transfer we should expect the surface waves, if they exist, to predominate at great distances.

Jeffreys (1931), in the investigation mentioned above, attempted to recombine certain component pulses to demonstrate the interference and identify the resulting motion with the anticipated Love-wave phase; the present problem is so much more complicated that a similar procedure could not be contemplated.

Alternatively, we may look for a new transformation of the ζ- and ω-contours which will separate out that part of the disturbance which predominates at great distances. It will be

* I am indebted to Mr R. A. Brooker of the Mathematical Laboratory, Cambridge, who undertook the 'programming' of the problem and its solution on the E.D.S.A.C.

found that such a transformation is possible and that we do, in fact, obtain as the dominant motion a certain superposition of the free waves which may be propagated over the surface of the elastic system. We shall first investigate these free-surface waves without reference to the generating pulse.

PART II. THE DISPERSIVE SURFACE WAVE-TRAIN GENERATED BY THE LINE SOURCE

14. The propagation of free-surface waves in a semi-infinite elastic medium with a single surface layer

Consider the elastic system consisting of a semi-infinite medium of density ρ_2, elastic constants λ_2, μ_2 on which lies a finite depth H of material of density ρ_1 and elastic constants λ_1, μ_1.

Defining right-handed axes as shown, it is desired to investigate the two-dimensional vibrations of the system in the plane (Ox, Oz), propagated over the surface without change of form and vanishing at great depths.

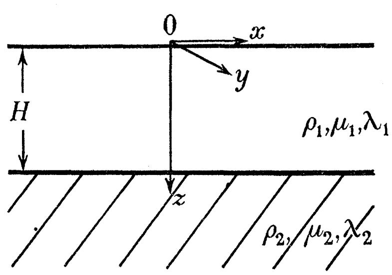

Figure 25

This problem has been considered by a number of authors who for various ratios of density and elastic constants have shown that such a motion is possible and obtained the relation between phase-velocity and period in the first mode of vibration. In particular, Lee (1935) and Jeffreys (1935) obtained the wave-velocity-period curve for the model earth considered in this paper, that of a granitic layer overlying a great depth of ultra-basic rock, described roughly by

$$\left.\begin{aligned} \mu_2/\mu_1 = \lambda_2/\lambda_1 = \tfrac{20}{9}, \\ \rho_2/\rho_1 = \tfrac{5}{4}, \quad \mu_1 = \lambda_1, \quad \mu_2 = \lambda_2. \end{aligned}\right\} \tag{14·1}$$

Although it has long been recognized that the familiar velocity-period curve represents only one solution of the wave-velocity equation, and that, as for Love waves, there might exist an infinite number of modes of vibration, no systematic investigation of the higher modes appears to have been made.*

The importance of a knowledge of all the modes of vibration will be evident when we refer again later to the generating pulse, so that the work of Lee and Jeffreys will now be extended.

* For a liquid/solid system the corresponding study has been made by Press, Ewing & Tolstoy (1950). See also Longuet-Higgins (1950).

As the elementary theory of elastic waves has been presented in part I, § 2, the potentials ϕ and ψ and velocities α and β defined and the boundary conditions postulated for just such a medium, we shall introduce without further justification solutions ϕ and ψ of the form

$$\left.\begin{array}{ll}\text{upper layer} & \begin{cases}\phi = [A\exp\{-(z-H)\lambda_{\alpha_1}\}+B\exp\{(z-H)\lambda_{\alpha_1}\}]\cos\kappa(x-ct),\\ \psi = [C\sinh(z-H)\lambda_{\beta_1}+D\cosh(z-H)\lambda_{\beta_1}]\sin\kappa(x-ct),\end{cases}\\ \text{lower layer} & \begin{cases}\phi = R\exp\{-(z-H)\lambda_{\alpha_2}\}\cos\kappa(x-ct),\\ \psi = Q\exp\{-(z-H)\lambda_{\beta_2}\}\sin\kappa(x-ct),\end{cases}\end{array}\right\}\quad (c<\beta_1), \tag{14·2}$$

where

$$\left.\begin{aligned}\lambda_{\beta_1} &= (\kappa^2-\kappa^2_{\beta_1})^{\frac{1}{2}} = \kappa(1-c^2/\beta_1^2)^{\frac{1}{2}},\\ \lambda_{\alpha_1} &= (\kappa^2-\kappa^2_{\alpha_1})^{\frac{1}{2}} = \kappa(1-c^2/\alpha_1^2)^{\frac{1}{2}},\\ \lambda_{\beta_2} &= (\kappa^2-\kappa^2_{\beta_2})^{\frac{1}{2}} = \kappa(1-c^2/\beta_2^2)^{\frac{1}{2}},\\ \lambda_{\alpha_2} &= (\kappa^2-\kappa^2_{\alpha_2})^{\frac{1}{2}} = \kappa(1-c^2/\alpha_2^2)^{\frac{1}{2}},\end{aligned}\right\} \tag{14·3}$$

and

$$\left.\begin{array}{ll}\text{upper layer} & \begin{cases}\phi = [A\exp\{-(z-H)\lambda_{\alpha_1}\}+B\exp\{(z-H)\lambda_{\alpha_1}\}]\cos\kappa(x-ct),\\ \psi = [C\sin(z-H)\bar{\lambda}_{\beta_1}+D\cos(z-H)\bar{\lambda}_{\beta_1}]\sin\kappa(x-ct),\end{cases}\\ \text{lower layer} & \begin{cases}\phi = R\exp\{-(z-H)\lambda_{\alpha_2}\}\cos\kappa(x-ct),\\ \psi = Q\exp\{-(z-H)\lambda_{\beta_2}\}\sin\kappa(x-ct),\end{cases}\end{array}\right\}\quad (\beta_2>c>\beta_1), \tag{14·4}$$

where

$$\bar{\lambda}_{\beta_1} = (\kappa^2_{\beta_1}-\kappa^2)^{\frac{1}{2}} = \kappa(c^2/\beta_1^2-1)^{\frac{1}{2}}. \tag{14·5}$$

This solution for $\beta_2>c>\beta_1$ may be derived from that for $c<\beta_1$ putting $\bar{\lambda}_{\beta_1} = \mathrm{i}\lambda_{\beta_1}$.

From the factors $\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}\kappa(x-ct)$ it is seen that c is the phase-velocity, $2\pi/\kappa$ the wave-length and $2\pi/\omega$ $(\omega = \kappa c)$ the period of the disturbance. The condition of continuous propagation without change of form requires that κ be real, and the vanishing of u and w as $z\to\infty$ that λ_{α_2}, λ_{β_2} be real. Thus with the given ratios of elastic constants and density for which

$$\alpha_2>\alpha_1>\beta_2>\beta_1,$$

we are interested only in values of c less than β_2.

The boundary conditions of a free surface and continuous displacements and stresses at the interface may be written (for $c<\beta_1$)

$$\left.\begin{aligned}&-\kappa A-\kappa B-\lambda_{\beta_1}C = -\kappa R-\lambda_{\beta_2}Q,\\ &-\lambda_{\alpha_1}A+\lambda_{\alpha_1}B-\kappa D = -\lambda_{\alpha_2}R-\kappa Q,\\ &2\kappa\lambda_{\alpha_1}A-2\kappa\lambda_{\alpha_1}B+(2\kappa^2-\kappa^2_{\beta_1})D = 2\kappa\lambda_{\alpha_2}(\mu_2/\mu_1)R+(2\kappa^2-\kappa^2_{\beta_2})(\mu_2/\mu_1)Q,\\ &(2\kappa^2-\kappa^2_{\beta_1})A+(2\kappa^2-\kappa^2_{\beta_1})B+2\kappa\lambda_{\beta_1}C = (2\kappa^2-\kappa^2_{\beta_2})(\mu_2/\mu_1)R+2\kappa\lambda_{\beta_2}(\mu_2/\mu_1)Q,\\ &2\kappa\lambda_{\alpha_1}\exp\{H\lambda_{\alpha_1}\}A-2\kappa\lambda_{\alpha_1}\exp\{-H\lambda_{\alpha_1}\}B+(2\kappa^2-\kappa^2_{\beta_1})\{\sinh(H\lambda_{\beta_1})C+\cosh(H\lambda_{\beta_1})D\} = 0,\\ &(2\kappa^2-\kappa^2_{\beta_1})\exp\{H\lambda_{\alpha_1}\}A+(2\kappa^2-\kappa^2_{\beta_1})\exp\{-H\lambda_{\alpha_1}\}B+2\kappa\lambda_{\beta_1}\{\cosh(H\lambda_{\beta_1})C+\sinh(H\lambda_{\beta_1})D\} = 0,\end{aligned}\right\} \tag{14·6}$$

with a similar set of equations for $\beta_2>c>\beta_1$. These have a solution if the determinant of the coefficients of A, B, C, D, R, Q is zero, that is,

$c<\beta_1$:

$$\begin{aligned}\Delta = 0 = \;& \exp\{H\lambda_{\alpha_1}\}\,[-4\kappa^2\lambda_{\alpha_1}\lambda_{\beta_1}\cosh H\lambda_{\beta_1}+(2\kappa^2-\kappa_{\beta_1}^2)^2\sinh H\lambda_{\beta_1}]\,S\\ &+\exp\{H\lambda_{\alpha_1}\}\,[4\kappa^2\lambda_{\alpha_1}\lambda_{\beta_1}\sinh H\lambda_{\beta_1}-(2\kappa^2-\kappa_{\beta_1}^2)^2\cosh H\lambda_{\beta_1}]\,T\\ &+[4\kappa\lambda_{\alpha_1}(2\kappa^2-\kappa_{\beta_1}^2)\,V-2\kappa\lambda_{\beta_1}(2\kappa^2-\kappa_{\beta_1}^2)\,Y]\\ &+\exp\{-H\lambda_{\alpha_1}\}\,[-4\kappa^2\lambda_{\alpha_1}\lambda_{\beta_1}\cosh H\lambda_{\beta_1}-(2\kappa^2-\kappa_{\beta_1}^2)^2\sinh H\lambda_{\beta_1}]\,W\\ &+\exp\{-H\lambda_{\alpha_1}\}\,[4\kappa^2\lambda_{\alpha_1}\lambda_{\beta_1}\sinh H\lambda_{\beta_1}+(2\kappa^2-\kappa_{\beta_1}^2)^2\cosh H\lambda_{\beta_1}]\,U. \qquad (14\cdot7)\end{aligned}$$

$\beta_2>c>\beta_1$:

$$\begin{aligned}\Delta = 0 = \;& \exp\{H\lambda_{\alpha_1}\}\,[4\kappa^2\lambda_{\alpha_1}\bar\lambda_{\beta_1}\cos H\bar\lambda_{\beta_1}-(2\kappa^2-\kappa_{\beta_1}^2)^2\sin H\bar\lambda_{\beta_1}]\,S\\ &+\exp\{H\lambda_{\alpha_1}\}\,[-4\kappa^2\lambda_{\alpha_1}\bar\lambda_{\beta_1}\sin H\bar\lambda_{\beta_1}-(2\kappa^2-\kappa_{\beta_1}^2)^2\cos H\bar\lambda_{\beta_1}]\,T\\ &+[4\kappa\lambda_{\alpha_1}(2\kappa^2-\kappa_{\beta_1}^2)\,V+2\kappa\bar\lambda_{\beta_1}(2\kappa^2-\kappa_{\beta_1}^2)^2\,Y]\\ &+\exp\{-H\lambda_{\alpha_1}\}\,[4\kappa^2\lambda_{\alpha_1}\bar\lambda_{\beta_1}\cos H\bar\lambda_{\beta_1}+(2\kappa^2-\kappa_{\beta_1}^2)^2\sin H\bar\lambda_{\beta_1}]\,W\\ &+\exp\{-H\lambda_{\alpha_1}\}\,[-4\kappa^2\lambda_{\alpha_1}\bar\lambda_{\beta_1}\sin H\bar\lambda_{\beta_1}+(2\kappa^2-\kappa_{\beta_1}^2)^2\cos H\bar\lambda_{\beta_1}]\,U, \qquad (14\cdot8)\end{aligned}$$

where S, T, W, U, V, Y are the 4×4 determinants.

$c<\beta_1$:

$$\left.\begin{aligned}
S &= \begin{vmatrix} -\kappa & 0 & -\kappa & -\lambda_{\beta_2}\\ \lambda_{\alpha_1} & -\kappa & -\lambda_{\alpha_2} & -\kappa\\ -2\kappa\lambda_{\alpha_1} & (2\kappa^2-\kappa_{\beta_1}^2) & 2\kappa\lambda_{\alpha_2}\mu_2/\mu_1 & (2\kappa^2-\kappa_{\beta_2}^2)\,\mu_2/\mu_1\\ (2\kappa^2-\kappa_{\beta_1}^2) & 0 & (2\kappa^2-\kappa_{\beta_2}^2)\,\mu_2/\mu_1 & 2\kappa\lambda_{\beta_2}\mu_2/\mu_1\end{vmatrix};\\
T &= \begin{vmatrix} -\kappa & -\lambda_{\beta_1} & . & .\\ \lambda_{\alpha_1} & 0 & . & .\\ -2\kappa\lambda_{\alpha_1} & 0 & . & .\\ (2\kappa^2-\kappa_{\beta_1}^2) & 2\kappa\lambda_{\beta_1} & . & .\end{vmatrix}; \quad
W = \begin{vmatrix} -\kappa & 0 & . & .\\ -\lambda_{\alpha_1} & -\kappa & . & .\\ 2\kappa\lambda_{\alpha_1} & (2\kappa^2-\kappa_{\beta_1}^2) & . & .\\ (2\kappa^2-\kappa_{\beta_1}^2) & 0 & . & .\end{vmatrix};\\
U &= \begin{vmatrix} -\kappa & -\lambda_{\beta_1} & . & .\\ -\lambda_{\alpha_1} & 0 & . & .\\ 2\kappa\lambda_{\alpha_1} & 0 & . & .\\ (2\kappa^2-\kappa_{\beta_1}^2) & 2\kappa\lambda_{\beta_1} & . & .\end{vmatrix}; \quad
V = \begin{vmatrix} -\lambda_{\beta_1} & 0 & . & .\\ 0 & -\kappa & . & .\\ 0 & (2\kappa^2-\kappa_{\beta_1}^2) & . & .\\ 2\kappa\lambda_{\beta_1} & 0 & . & .\end{vmatrix};\\
Y &= \begin{vmatrix} -\kappa & -\kappa & . & .\\ -\lambda_{\alpha_1} & \lambda_{\alpha_1} & . & .\\ 2\kappa\lambda_{\alpha_1} & -2\kappa\lambda_{\alpha_1} & . & .\\ (2\kappa^2-\kappa_{\beta_1}^2) & (2\kappa^2-\kappa_{\beta_1}^2) & . & .\end{vmatrix}
\end{aligned}\right\} \qquad (14\cdot9)$$

(the 3rd and 4th columns of $T, \ldots, Y$ are identical with those of S). When $\beta_2>c>\beta_1$ they are the corresponding forms obtained replacing λ_{β_1} in (14·9) by $-\bar\lambda_{\beta_1}$.

Equations (14·7) and (14·8) are the 'wave equations' determining the period $2\pi/\kappa c$ as a function of the phase-velocity c. They can only be solved numerically and it is usual to give c a series of values and solve for κ. From the form of (14·8) and the multivaluedness of the functions $\cos^{-1}$ and $\sin^{-1}$ it appears that κ might be a multivalued function of c in $\beta_2 > c > \beta_1$. When κH is large, an approximation to (14·8) is

$$\tan\{\kappa H (c^2/\beta_1^2 - 1)^{\frac{1}{2}}\} \doteqdot \frac{4(c^2/\beta_1^2-1)^{\frac{1}{2}}(1-c^2/\alpha_1^2)^{\frac{1}{2}} S - (2-c^2/\beta_1^2)^2 T}{4(c^2/\beta_1^2-1)^{\frac{1}{2}}(1-c^2/\alpha_1^2)^{\frac{1}{2}} T + (2-c^2/\beta_1^2)^2 S} \tag{14·10}$$

$$= \tan\theta_c, \text{ say } \quad (\tfrac{1}{2}\pi > \theta_c > -\tfrac{1}{2}\pi),$$

or

$$\kappa H \doteqdot \frac{\theta_c + (n-1)\pi}{(c^2/\beta_1^2-1)^{\frac{1}{2}}} \quad (n = 1, 2, \ldots), \tag{14·11}$$

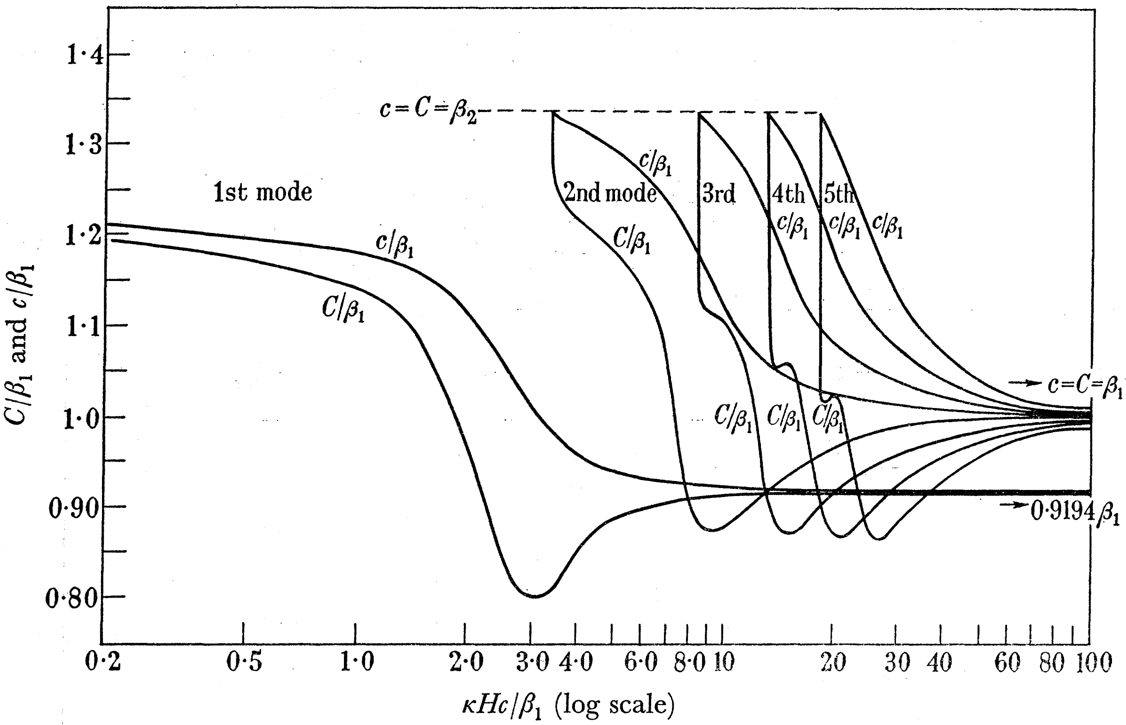

FIGURE 26. Phase- and group-velocity of Rayleigh waves in the first five modes of vibration.

$$\left.\begin{aligned} \rho_2/\rho_1 &= 5/4 \\ \mu_2/\mu_1 &= 20/9 \\ \lambda &= \mu \end{aligned}\right\} \begin{aligned} \alpha_1 &= \sqrt{3}\,\beta_1 \\ \alpha_2 &= \sqrt{3}\,\beta_2 \\ \beta_2 &= 4/3\,\beta_1. \end{aligned}$$

and there is clearly an infinity of solutions. Numerical computation shows this to be so throughout the range $\beta_2 > c > \beta_1$, each value corresponding to a distinct mode of propagation. For $c < \beta_1$, (14·7) gives a single value of κ corresponding to a part of the first mode.

Values of κH were determined for the first five modes for c/β_1 at intervals of 0·01 (finer intervals near points of inflexion and stationary points) and the results are tabulated below.

In figure 26 c/β_1 is graphed against $\kappa Hc/\beta_1$, the dimensionless quantity inversely proportional to the period. For short period (and therefore wave-length) the phase-velocity in the first mode approaches that of ordinary Rayleigh waves in medium I and for long period that in the lower medium, II. The second and higher modes are characterized by a cut-off period (which decreases for successive modes) corresponding to the phase-velocity

β_2 and above which no such surface motion occurs. In each, the period becomes infinitely short as $c \to \beta_1$.

The approximation (14·10) for κH large (short periods) was found useful in obtaining a first approximation to solutions in the second and higher modes. On account of the

TABLE 1

c	1st mode κH	2nd mode κH	3rd mode κH	4th mode κH	5th mode κH
1·333	—	2·455	6·022	9·587	13·15
1·330	—	2·631	6·208	9·791	13·37
1·32	—	2·915	6·543	10·19	13·83
1·31	—	3·198	6·881	10·59	14·30
1·30	—	3·508	7·252	11·03	14·81
1·29	—	3·848	7·660	11·51	15·36
1·28	—	4·199	8·091	12·02	15·95
1·27	—	4·549	8·518	12·53	16·54
1·26	—	4·885	8·943	13·04	17·14
1·25	—	5·192	9·343	13·53	17·72
1·24	—	5·473	9·726	14·01	18·29
1·23	($c = 1·2259... \to \kappa H = 0$)	5·733	10·09	14·48	18·87
1·22	0·062	5·979	10·45	14·95	19·44
1·21	0·181	6·218	10·81	15·42	20·04
1·20	0·339	6·452	11·17	15·91	20·65
1·19	0·534	6·684	11·54	16·41	21·28
1·18	0·746	6·920	11·93	16·94	21·96
1·17	0·987	7·164	12·33	17·50	22·67
1·16	1·175	7·418	12·75	18·10	23·44
1·15	1·333	7·684	13·21	18·74	24·27
1·14	1·468	7·968	13·70	19·44	25·18
1·13	1·598	8·274	14·24	20·21	26·18
1·12	1·711	8·607	14·83	21·06	27·29
1·11	1·821	8·974	15·49	22·01	28·53
1·10	1·924	9·383	16·24	23·09	29·95
1·09	2·026	9·849	17·09	24·34	31·58
1·08	2·127	10·385	18·09	25·79	33·49
1·07	2·228	11·02	19·21	27·52	35·77
1·06	2·332	11·78	20·72	29·65	38·59
1·05	2·437	12·75	22·56	32·37	42·18
1·04	2·548	14·02	25·02	36·02	47·01
1·03	2·665	15·86	28·59	41·32	54·05
1·02	2·788	18·86	34·49	50·12	65·75
1·01	2·922	25·51	47·66	69·81	91·97
1·00	3·069	∞	∞	∞	∞
0·99	3·233				
0·98	3·421				
0·97	3·638				
0·96	3·918				
0·95	4·249				
0·94	4·724				
0·93	5·707				
0·92	12·3				
0·9194...	∞				

decreasing 'cut-off period' it becomes more accurate over a greater range of c as higher modes are considered.

In the first mode $\kappa H \to \infty$ as $c \to 0·9194\ldots\beta_1$, and the corresponding approximation for short periods as derived from equation (14·7) is

$$\exp\{-2\kappa H(1-c^2/\beta_1^2)^{\frac{1}{2}}\} \doteqdot \frac{[(2-c^2/\beta_1^2)^2-4\,(1-c^2/\beta_1^2)^{\frac{1}{2}}\,(1-c^2/\alpha_1^2)^{\frac{1}{2}}]\,(S-T)}{[(2-c^2/\beta_1^2)^2+4\,(1-c^2/\beta_1^2)^{\frac{1}{2}}\,(1-c^2/\alpha_1^2)^{\frac{1}{2}}]\,(S+T)}. \quad (14·12)$$

The group-velocity was calculated by numerical differentiation of the tables of κH against phase-velocity c, according to the formulae

$$\text{(group velocity) } C = \frac{\mathrm{d}}{\mathrm{d}\kappa}(\kappa c) = c + \kappa H \frac{\mathrm{d}c}{\mathrm{d}(\kappa H)}, \tag{14·13}$$

$$\left(\frac{\mathrm{d}(\kappa H)}{\mathrm{d}c}\right)_0 \simeq \frac{1}{\delta c}\left[\Delta_0'(\kappa H) - \tfrac{1}{6}\Delta_0'''(\kappa H)\right], \quad \begin{cases} \Delta_0' = \frac{1}{2}(\Delta_{-\frac{1}{2}}' + \Delta_{\frac{1}{2}}') \\ \Delta_0''' = \frac{1}{2}(\Delta_{-\frac{1}{2}}''' + \Delta_{\frac{1}{2}}''') \end{cases} \tag{14·14}$$

with the standard 'difference' notation and where κH is given at uniform intervals of c. As the period decreases κH changes so rapidly that numerical differentiation is inadequate, but this is just the condition under which the approximations (14·10) and (14·12) hold. Differentiation of these gives useful expressions for $\mathrm{d}c/\mathrm{d}(\kappa H)$ as $\kappa H \to \infty$.

In the second and higher modes we have

$$\kappa H + [\tan^{-1}(T/S) - (n-1)\pi]/[c^2/\beta_1^2 - 1]^{\frac{1}{2}} \doteqdot 0,$$

yielding

$$\frac{\mathrm{d}c}{\mathrm{d}(\kappa H)}_{\substack{\kappa H \to \infty \\ c \to \beta_1}} \doteqdot \left[-\frac{c}{\beta_1^2}\frac{\tan^{-1}(T/S) - (n-1)\pi}{(c^2/\beta_1^2 - 1)^{\frac{3}{2}}} + \frac{1}{(1+T^2/S^2)(c^2/\beta_1^2 - 1)^{\frac{1}{2}}}\frac{\mathrm{d}(T/S)}{\mathrm{d}c}\right]^{-1}, \tag{14·15}$$

and using the fact that

$$T/S = \bar{\lambda}_{\beta_1} \times (\text{a factor which does not vanish as } c \to \beta_1),$$

this gives

$$\frac{\mathrm{d}c}{\mathrm{d}(\kappa H)} \doteqdot \left[\frac{\kappa H c}{\beta_1^2(c^2/\beta_1^2 - 1)} + \frac{T}{S}\frac{c}{\beta_1^2(c^2/\beta_1^2 - 1)^{\frac{3}{2}}}\right]^{-1}$$

$$\doteqdot [\kappa H c/\beta_1^2(c^2/\beta_1^2 - 1)]^{-1}, \tag{14·16}$$

or

$$\kappa\frac{\mathrm{d}c}{\mathrm{d}\kappa} \doteqdot (c^2 - \beta_1^2)/c, \quad \text{independent of mode (to the first order).}$$

In the first mode

$$\frac{\mathrm{d}c}{\mathrm{d}(\kappa H)}_{\substack{\kappa H = \infty \\ c \to 0\cdot9194\ldots\beta_1}}$$

$$\doteqdot \frac{2(1-c^2/\beta_1^2)^{\frac{1}{2}}[(2-c^2/\beta_1^2)^2 + 4(1-c^2/\beta_1^2)^{\frac{1}{2}}(1-c^2/\alpha_1^2)^{\frac{1}{2}}](S+T)}{[-4c(2-2c^2/\beta_1^2)/\beta_1^2 + 4c(1-c^2/\alpha_1^2)^{\frac{1}{2}}/\beta_1^2(1-c^2/\beta_1^2)^{\frac{1}{2}} + 4c(1-c^2/\beta_1^2)^{\frac{1}{2}}/\alpha_1^2(1-c^2/\alpha_1^2)^{\frac{1}{2}}](S-T)}$$

$$\times \exp\{-2\kappa H(1-c^2/\beta_1^2)^{\frac{1}{2}}\}, \tag{14·17}$$

and as the period shortens the factor $\kappa H \dfrac{\mathrm{d}c}{\mathrm{d}(\kappa H)}$ tends to zero like

$$\kappa H \exp\{-2\kappa H(1-0\cdot9194\ldots^2)^{\frac{1}{2}}\}.$$

It is seen that a pronounced minimum group-velocity is a feature of all modes, occurring for periods given by

$$\kappa H = 2\cdot92\ldots, \quad 7\cdot97\ldots, \quad 12\cdot75\ldots \quad (n = 1, 2, 3, \ldots),$$

and increasing from a value $0\cdot801\ldots\beta_1$ in the first mode to $0\cdot866\ldots\beta_1$ in the fourth and thereafter apparently decreasing slowly but steadily. In addition, there is a conspicuous

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

feature occurring for relatively shorter periods which changes from an inflexion in the first mode to a definite double 'kink' for higher modes. It appears that although the maximum and minimum become sharper the 'span' of the kink contracts. It seems reasonable to suppose that as n becomes very great the peaks will coincide and the curve be smoothed in that region.

In anticipation of their use in the application of these results to the problem of the generating pulse, values of $\mathrm{d}C/\mathrm{d}(\kappa c)$, $\mathrm{d}^2 C/\mathrm{d}(\kappa c)^2$, etc., were obtained numerically.

In an ideal system these surface waves, once generated, are capable of continuous propagation. We shall now return to the problem of the cylindrical pulse and attempt to determine the relative excitation of each mode by that pulse.

15. The surface-wave motion due to the line source

We return now to the problem of the line source. For an initial P-pulse the motion in the surface-layer is given by

$$\Phi_p = \Phi_0 + \frac{1}{2\pi \mathrm{i}} \int_\Omega \frac{\mathrm{d}\omega}{\omega} \int_{-\infty}^{\infty} \left[4\frac{\Delta_A}{\Delta_p} \exp\{-(z-H)\lambda_{\alpha_1}\} + 4\frac{\Delta_B}{\Delta_p} \exp\{(z-H)\lambda_{\alpha_1}\} \right] \mathrm{e}^{\mathrm{i}\omega t \mp \mathrm{i}\zeta x} \mathrm{d}\zeta \quad (\mathscr{R}(\omega) \gtrless 0), \tag{15·1}$$

$$\Psi_p = \frac{1}{2\pi \mathrm{i}} \int_\Omega \frac{\mathrm{d}\omega}{\omega} \int_{-\infty}^{\infty} \left[4\frac{\Delta_C}{\Delta_p} \exp\{-(z-H)\lambda_{\beta_1}\} + 4\frac{\Delta_D}{\Delta_p} \exp\{(z-H)\lambda_{\beta_1}\} \right] \mathrm{e}^{\mathrm{i}\omega t \mp \mathrm{i}\zeta x} \mathrm{d}\zeta \quad (\mathscr{R}(\omega) \gtrless 0), \tag{15·2}$$

where Φ_0, Δ_A, ..., etc., together with the expressions for Φ and Ψ in the lower medium, were given in part I, § 3.

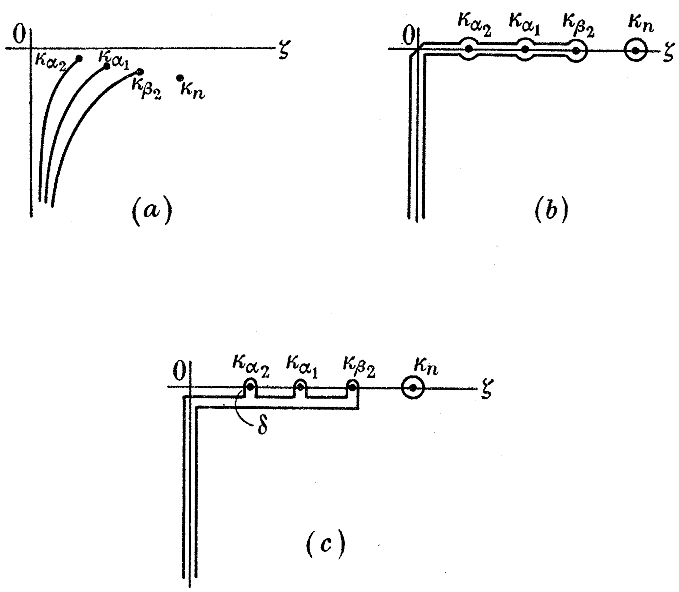

Figure 27. Limiting forms of the contour Γ as the Ω contour approaches the real axis.

If we suppose that the Ω-contour approaches the real axis from below, then we must consider the ζ-integrals for the limiting case of ω real. We saw that for $|\omega| > 0$ each of the cuts $\mathscr{R}(\lambda_{\alpha_2}) = 0$, etc., was part of a hyperbola in the fourth quadrant which now approaches the bounding axes (figure 27 a). Since $\mathscr{R}(\lambda_{\alpha_2}) = 0$, $\mathscr{R}(\lambda_{\alpha_1}) = 0$, $\mathscr{R}(\lambda_{\beta_2}) = 0$ only are singular lines with respect to the integrands (15·1) and (15·2), we may in the limit replace the original ζ-contour by a single loop Γ_0 surrounding κ_{α_2}, κ_{α_1}, κ_{β_2} on the real axis together with small circles surrounding the poles outside Γ_0. By considering Γ_0 slightly deformed as in figure 27*c*, where the length δ is small but finite, it is seen that for large x, by virtue of the factor $e^{-i\zeta x}$, the main contribution from Γ_0 must come from the small portions surrounding the branch-points. Further, it is easily shown that these contributions decrease with distance at least as fast as $x^{-\frac{3}{2}}$. We shall go on to demonstrate that the contributions from the poles, on the other hand, decrease like $x^{-\frac{1}{2}}$ or less rapidly, so that they must predominate at great distances. It is on the basis of similar arguments that Pekeris (1948), Press *et al.* (1950) and Longuet-Higgins (1950) neglect entirely the branch-line integrals when considering 'liquid-liquid' and 'liquid-solid' systems respectively.

When $\omega<0$ the hyperbolas $\mathscr{R}(\lambda_{\alpha_2}) = 0$, etc., approach the bounding axes from within the first quadrant, but the conclusions reached are identical.

We shall now consider in detail that part of the disturbance contributed by the poles. Since Φ_0 is regular everywhere the poles of the integrands (15·1) and (15·2) are the zeros of Δ_p which we may denote by $\kappa_n(\omega)$. Thus at great distances we have approximately

upper layer

$$\phi = \frac{1}{2}\int_{\Sigma\Gamma_n}\left[4\frac{\Delta_A}{\Delta_p}\exp\{-(z-H)\lambda_{\alpha_1}\}+4\frac{\Delta_B}{\Delta_p}\exp\{(z-H)\lambda_{\alpha_1}\}\right]e^{i\omega t\mp i\zeta x}\,d\zeta \quad (\mathscr{R}(\omega)\gtrless 0), \tag{15·3}$$

$$\psi = \mp\frac{1}{2i}\int_{\Sigma\Gamma_n}\left[4\frac{\Delta_C}{\Delta_p}\exp\{-(z-H)\lambda_{\beta_1}\}+4\frac{\Delta_D}{\Delta_p}\exp\{(z-H)\lambda_{\beta_1}\}\right]e^{i\omega t\mp i\zeta x}\,d\zeta \quad (\mathscr{R}(\omega)\gtrless 0), \tag{15·4}$$

these being the solutions appropriate to steady-state propagation of harmonic compressional waves period $2\pi/\omega$ from the line source. Generalizing to the unit pulse it follows that

upper layer

$$\Phi = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{d\omega}{\omega}\int_{\Sigma\Gamma_n}\frac{1}{2}\left[4\frac{\Delta_A}{\Delta_p}\exp\{-(z-H)\lambda_{\alpha_1}\}+4\frac{\Delta_B}{\Delta_p}\exp\{(z-H)\lambda_{\alpha_1}\}\right]e^{i\omega t\mp i\zeta x}\,d\zeta, \tag{15·5}$$

$$\Psi = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{d\omega}{\omega}\int_{\Sigma\Gamma_n}\mp\frac{1}{2i}\left[4\frac{\Delta_C}{\Delta_p}\exp\{-(z-H)\lambda_{\beta_1}\}+4\frac{\Delta_D}{\Delta_p}\exp\{(z-H)\lambda_{\beta_1}\}\right]e^{i\omega t\mp i\zeta x}\,d\zeta, \tag{15·6}$$

with similar expressions for ϕ, ψ, Φ, Ψ in the lower medium. As this investigation is primarily aimed at explaining the complexities of seismograms obtained at or near the earth's surface, these latter expressions will not be discussed further.

In this consideration, ω is real and the factor $e^{-i\zeta x}$ shows that wave systems with ζ-complex will be attenuated exponentially with respect to x. We therefore are interested only in poles lying on the real axis and such that $\zeta>\kappa_{\beta_2}$. Since Δ_p is seen to be identical with the determinantal expression Δ arising in the discussion of the free-surface waves, the problem of determining the poles has been solved and the relation between the two studies becomes apparent.

To standardize the notation we shall henceforth write κ for ζ (the use of ζ in the earlier theory was prompted by the need for a symbol conveniently written in two parts, real and imaginary). As the Δ_s occurring in the problem of the initial *S*-pulse is also identical with Δ the suffixes p and s will be suppressed.

Using the theory of residues, (15·5) and (15·6) reduce to

$$\Phi = \int_{-\infty}^{\infty} \frac{d\omega}{\omega} \tfrac{1}{2} \sum_n \left[4\frac{\Delta_A}{\Delta'} \exp\{-(z-H)\lambda_{\alpha_1}\} + 4\frac{\Delta_B}{\Delta'} \exp\{(z-H)\lambda_{\alpha_1}\} \right]_{\kappa=\kappa_n} \exp\{i\omega t \mp i\kappa_n x\} \quad (\mathscr{R}(\omega) \gtrless 0), \qquad (15\cdot7)$$

$$\Psi = \int_{-\infty}^{\infty} \frac{d\omega}{\omega} \tfrac{1}{2} \sum_n \mp \left[4\frac{\Delta_C}{\Delta'} \exp\{-(z-H)\lambda_{\beta_1}\} + 4\frac{\Delta_D}{\Delta'} \exp\{(z-H)\lambda_{\beta_1}\} \right]_{\kappa=\kappa_n} \exp\{i\omega t \mp i\kappa_n x\} \quad (\mathscr{R}(\omega) \gtrless 0), \qquad (15\cdot8)$$

where

$$\Delta' = \left(\frac{\partial \Delta}{\partial \kappa}\right)_\omega, \qquad (15\cdot9)$$

and the summation extends over all the poles κ_n of $\Delta = 0$ for each value of ω. The fact that the many solutions κ_n for any one ω correspond to distinct modes of propagation (i.e. κ_n, for a given n, varies continuously with ω) permits the order of summation and integration to be reversed and (15·7) and (15·8) written

$$\Phi = \sum_n \int_{-\infty}^{\infty} \tfrac{1}{2}\bar{\phi}_n \exp\{i(\omega t \mp \kappa_n x)\} \frac{d\omega}{\omega} \qquad (\mathscr{R}(\omega) \gtrless 0), \qquad (15\cdot10)$$

$$\Psi = \sum_n \int_{-\infty}^{\infty} (\mp\tfrac{1}{2})\, \bar{\psi}_n \exp\{i(\omega t \mp \kappa_n x)\} \frac{d\omega}{\omega} \quad (\mathscr{R}(\omega) \gtrless 0), \qquad (15\cdot11)$$

where $\bar{\phi}_n$ is the expression

$$[(4\Delta_A/\Delta') \exp\{-(z-H)\lambda_{\alpha_1}\} + (4\Delta_B/\Delta') \exp\{(z-H)\lambda_{\alpha_1}\}]$$

expressed as a function of ω at a pole $\kappa_n(\omega)$; similarly $\bar{\psi}_n$. Thus

$c < \beta_1$:

$$\begin{aligned}
\Delta'\bar{\phi} = {} & 4\exp\{-(h+z)\lambda_{\alpha_1}\}\{4\zeta^2\lambda_{\beta_1}\exp\{H\lambda_{\alpha_1}\}(S\cosh H\lambda_{\beta_1} - T\sinh H\lambda_{\beta_1}) \\
& \quad - 2\zeta(2\zeta^2-\kappa_{\beta_1}^2)V + \zeta\lambda_{\beta_1}(2\zeta^2-\kappa_{\beta_1}^2)Y/\lambda_{\alpha_1} \\
& \quad + \tfrac{1}{2}\exp\{-H\lambda_{\alpha_1}\}\cosh H\lambda_{\beta_1}[4\zeta^2\lambda_{\beta_1}W - (2\zeta^2-\kappa_{\beta_1}^2)^2 U/\lambda_{\alpha_1}] \\
& \quad + \tfrac{1}{2}\exp\{-H\lambda_{\alpha_1}\}\sinh H\lambda_{\beta_1}[-4\zeta^2\lambda_{\beta_1}U + (2\zeta^2-\kappa_{\beta_1}^2)^2 W\lambda_{\alpha_1}]\} \\
& + 4\exp\{-(z-h)\lambda_{\alpha_1}\}\{-\zeta\lambda_{\beta_1}(2\zeta^2-\kappa_{\beta_1}^2)Y/\lambda_{\alpha_1} \\
& \quad + \tfrac{1}{2}\exp\{-H\lambda_{\alpha_1}\}\cosh H\lambda_{\beta_1}[-4\zeta^2\lambda_{\beta_1}W + (2\zeta^2-\kappa_{\beta_1}^2)^2 U/\lambda_{\alpha_1}] \\
& \quad + \tfrac{1}{2}\exp\{-H\lambda_{\alpha_1}\}\sinh H\lambda_{\beta_1}[4\zeta^2\lambda_{\beta_1}U - (2\zeta^2-\kappa_{\beta_1}^2)^2 W/\lambda_{\alpha_1}]\} \\
& + 4\exp\{(z-h)\lambda_{\alpha_1}\} \quad \{4\zeta^2\lambda_{\beta_1}\exp\{-H\lambda_{\alpha_1}\}(-W\cosh H\lambda_{\beta_1} + U\sinh H\lambda_{\beta_1}) \\
& \quad + 2\zeta(2\zeta^2-\kappa_{\beta_1}^2)V \\
& \quad + \tfrac{1}{2}\exp\{-H\lambda_{\alpha_1}\}\cosh H\lambda_{\beta_1}[4\zeta^2\lambda_{\beta_1}W + (2\zeta^2-\kappa_{\beta_1}^2)^2 U/\lambda_{\alpha_1}] \\
& \quad + \tfrac{1}{2}\exp\{-H\lambda_{\alpha_1}\}\sinh H\lambda_{\beta_1}[-4\zeta^2\lambda_{\beta_1}U - (2\zeta^2-\kappa_{\beta_1}^2)^2 W/\lambda_{\alpha_1}]\} \\
& + 4\exp\{(h+z)\lambda_{\alpha_1}\} \quad \{\tfrac{1}{2}\exp\{-H\lambda_{\alpha_1}\}\cosh H\lambda_{\beta_1}[-4\zeta^2\lambda_{\beta_1}W - (2\zeta^2-\kappa_{\beta_1}^2)^2 U/\lambda_{\alpha_1}] \\
& \quad + \tfrac{1}{2}\exp\{-H\lambda_{\alpha_1}\}\sinh H\lambda_{\beta_1}[4\zeta^2\lambda_{\beta_1}U + (2\zeta^2-\kappa_{\beta_1}^2)^2 W/\lambda_{\alpha_1}]\};
\end{aligned} \qquad (15\cdot12)$$

$$\begin{aligned}
\Delta'\bar{\psi} = {} & 4\sinh{(H-z)}\,\lambda_{\beta_1}\exp\{-h\lambda_{\alpha_1}\} \quad \{2\zeta(2\zeta^2-\kappa_{\beta_1}^2)\,(\exp\{-H\lambda_{\alpha_1}\}\,W-\exp\{H\lambda_{\alpha_1}\}\,S) \\
& \quad -4\zeta^2\lambda_{\beta_1}\sinh H\lambda_{\beta_1}\,Y \\
& \quad +\tfrac{1}{2}Y[4\zeta^2\lambda_{\beta_1}\sinh H\lambda_{\beta_1}-(2\zeta^2-\kappa_{\beta_1}^2)^2\cosh{(H\lambda_{\beta_1})}/\lambda_{\alpha_1}] \\
& \quad +\tfrac{1}{2}W[-4\zeta(2\zeta^2-\kappa_{\beta_1}^2)\exp\{-H\lambda_{\alpha_1}\}]\} \\
& +4\sinh{(H-z)}\,\lambda_{\beta_1}\exp\{h\lambda_{\alpha_1}\} \quad \{\tfrac{1}{2}Y[-4\zeta^2\lambda_{\beta_1}\sinh H\lambda_{\beta_1}+(2\zeta^2-\kappa_{\beta_1}^2)^2\cosh{(H\lambda_{\beta_1})}/\lambda_{\alpha_1}] \\
& \quad +\tfrac{1}{2}W[4\zeta(2\zeta^2-\kappa_{\beta_1}^2)\exp\{-H\lambda_{\alpha_1}\}]\} \\
& +4\cosh{(H-z)}\,\lambda_{\beta_1}\exp\{-h\lambda_{\alpha_1}\}\{2\zeta(2\zeta^2-\kappa_{\beta_1}^2)\exp\{H\lambda_{\alpha_1}\}\,T \\
& \quad -2\zeta(2\zeta^2-\kappa_{\beta_1}^2)\exp\{-H\lambda_{\alpha_1}\}\,U \\
& \quad +4\zeta^2\lambda_{\beta_1}\cosh H\lambda_{\beta_1}\,Y \\
& \quad +\tfrac{1}{2}Y[-4\zeta^2\lambda_{\beta_1}\cosh H\lambda_{\beta_1}+(2\zeta^2-\kappa_{\beta_1}^2)^2\sinh{(H\lambda_{\beta_1})}/\lambda_{\alpha_1}] \\
& \quad +\tfrac{1}{2}U[4\zeta(2\zeta^2-\kappa_{\beta_1}^2)\exp\{-H\lambda_{\alpha_1}\}]\} \\
& +4\cosh{(H-z)}\,\lambda_{\beta_1}\exp\{h\lambda_{\alpha_1}\} \quad \{\tfrac{1}{2}Y[4\zeta^2\lambda_{\beta_1}\cosh H\lambda_{\beta_1}-(2\zeta^2-\kappa_{\beta_1}^2)^2\sinh{(H\lambda_{\beta_1})}/\lambda_{\alpha_1}] \\
& \quad +\tfrac{1}{2}U[-4\zeta(2\zeta^2-\kappa_{\beta_1}^2)\exp\{-H\lambda_{\alpha_1}\}]\}; \qquad (15\cdot13)
\end{aligned}$$

$\beta_2>c>\beta_1$:

$$\begin{aligned}
\Delta'\bar{\phi} = {} & 4\exp\{-(h+z)\,\lambda_{\alpha_1}\} \quad \{4\zeta^2\bar{\lambda}_{\beta_1}(-S\cos H\bar{\lambda}_{\beta_1}+T\sin H\bar{\lambda}_{\beta_1}) \\
& \quad -2\zeta(2\zeta^2-\kappa_{\beta_1}^2)\,V-\zeta\bar{\lambda}_{\beta_1}(2\zeta^2-\kappa_{\beta_1}^2)\,Y/\lambda_{\alpha_1} \\
& \quad +\tfrac{1}{2}\exp\{-H\lambda_{\alpha_1}\}\cos H\bar{\lambda}_{\beta_1}[-4\zeta^2\bar{\lambda}_{\beta_1}W-(2\zeta^2-\kappa_{\beta_1}^2)^2\,U/\lambda_{\alpha_1}] \\
& \quad +\tfrac{1}{2}\exp\{-H\lambda_{\alpha_1}\}\sin H\bar{\lambda}_{\beta_1}[4\zeta^2\bar{\lambda}_{\beta_1}\,U-(2\zeta^2-\kappa_{\beta_1}^2)^2\,W/\lambda_{\alpha_1}]\} \\
& +4\exp\{-(z-h)\,\lambda_{\alpha_1}\}\{\zeta\bar{\lambda}_{\beta_1}(2\zeta^2-\kappa_{\beta_1}^2)\,Y/\lambda_{\alpha_1} \\
& \quad +\tfrac{1}{2}\exp\{-H\lambda_{\alpha_1}\}\cos H\bar{\lambda}_{\beta_1}[4\zeta^2\bar{\lambda}_{\beta_1}W+(2\zeta^2-\kappa_{\beta_1}^2)^2\,U/\lambda_{\alpha_1}] \\
& \quad +\tfrac{1}{2}\exp\{-H\lambda_{\alpha_1}\}\sin H\bar{\lambda}_{\beta_1}[-4\zeta^2\bar{\lambda}_{\beta_1}\,U+(2\zeta^2-\kappa_{\beta_1}^2)^2\,W/\lambda_{\alpha_1}]\} \\
& +4\exp\{(z-h)\,\lambda_{\alpha_1}\} \quad \{4\zeta^2\bar{\lambda}_{\beta_1}\exp\{-H\lambda_{\alpha_1}\}\,(W\cos H\bar{\lambda}_{\beta_1}-U\sin H\bar{\lambda}_{\beta_1}) \\
& \quad +2\zeta(2\zeta^2-\kappa_{\beta_1}^2)\,V \\
& \quad +\tfrac{1}{2}\exp\{-H\lambda_{\alpha_1}\}\cos H\bar{\lambda}_{\beta_1}[-4\zeta^2\bar{\lambda}_{\beta_1}W+(2\zeta^2-\kappa_{\beta_1}^2)^2\,U/\lambda_{\alpha_1}] \\
& \quad +\tfrac{1}{2}\exp\{-H\lambda_{\alpha_1}\}\sin H\bar{\lambda}_{\beta_1}[4\zeta^2\bar{\lambda}_{\beta_1}\,U+(2\zeta^2-\kappa_{\beta_1}^2)^2\,W/\lambda_{\alpha_1}]\} \\
& +4\exp\{(z+h)\,\lambda_{\alpha_1}\} \quad \{\tfrac{1}{2}\exp\{-H\lambda_{\alpha_1}\}\cos H\bar{\lambda}_{\beta_1}[4\zeta^2\bar{\lambda}_{\beta_1}W-(2\zeta^2-\kappa_{\beta_1}^2)^2\,U/\lambda_{\alpha_1}] \\
& \quad +\tfrac{1}{2}\exp\{-H\lambda_{\alpha_1}\}\sin H\bar{\lambda}_{\beta_1}[-4\zeta^2\bar{\lambda}_{\beta_1}\,U-(2\zeta^2-\kappa_{\beta_1}^2)^2\,W/\lambda_{\alpha_1}]\}; \\
& \qquad (15\cdot14)
\end{aligned}$$

$$\begin{aligned}
\Delta'\bar{\psi} = {} & 4\sin{(z-H)}\,\bar{\lambda}_{\beta_1}\exp\{-h\lambda_{\alpha_1}\} \quad \{2\zeta(2\zeta^2-\kappa_{\beta_1}^2)\,(\exp\{-H\lambda_{\alpha_1}\}\,W-\exp\{H\lambda_{\alpha_1}\}\,S) \\
& \quad +4\zeta^2\bar{\lambda}_{\beta_1}\sin H\bar{\lambda}_{\beta_1}\,Y \\
& \quad +\tfrac{1}{2}Y[-4\zeta^2\bar{\lambda}_{\beta_1}\sin H\bar{\lambda}_{\beta_1}-(2\zeta^2-\kappa_{\beta_1}^2)^2\cos{(H\bar{\lambda}_{\beta_1})}/\lambda_{\alpha_1}] \\
& \quad +\tfrac{1}{2}W[-4\zeta(2\zeta^2-\kappa_{\beta_1}^2)]\exp\{-H\lambda_{\alpha_1}\}\} \\
& +4\sin{(z-H)}\,\bar{\lambda}_{\beta_1}\exp\{h\lambda_{\alpha_1}\} \quad \{\tfrac{1}{2}Y[4\zeta^2\bar{\lambda}_{\beta_1}\sin H\bar{\lambda}_{\beta_1}+(2\zeta^2-\kappa_{\beta_1}^2)^2\cos{(H\bar{\lambda}_{\beta_1})}/\lambda_{\alpha_1}] \\
& \quad +\tfrac{1}{2}W[4\zeta(2\zeta^2-\kappa_{\beta_1}^2)\exp\{-H\lambda_{\alpha_1}\}]\} \\
& +4\cos{(z-H)}\,\bar{\lambda}_{\beta_1}\exp\{-h\lambda_{\alpha_1}\}\{2\zeta(2\zeta^2-\kappa_{\beta_1}^2)\,(\exp\{H\lambda_{\alpha_1}\}\,T-\exp\{-H\lambda_{\alpha_1}\}\,U) \\
& \quad -4\zeta^2\lambda_{\beta_1}\cos H\bar{\lambda}_{\beta_1}\,Y \\
& \quad +\tfrac{1}{2}Y[4\zeta^2\bar{\lambda}_{\beta_1}\cos H\bar{\lambda}_{\beta_1}-(2\zeta^2-\kappa_{\beta_1}^2)^2\sin{(H\bar{\lambda}_{\beta_1})}/\lambda_{\alpha_1}] \\
& \quad +\tfrac{1}{2}U[4\zeta(2\zeta^2-\kappa_{\beta_1}^2)\exp\{-H\lambda_{\alpha_1}\}]\} \\
& +4\cos{(z-H)}\,\bar{\lambda}_{\beta_1}\exp\{h\lambda_{\alpha_1}\} \quad \{\tfrac{1}{2}Y[-4\zeta^2\lambda_{\beta_1}\cos H\bar{\lambda}_{\beta_1}+(2\zeta^2-\kappa_{\beta_1}^2)^2\sin{(H\bar{\lambda}_{\beta_1})}/\lambda_{\alpha_1}] \\
& \quad +\tfrac{1}{2}U[-4\zeta(2\zeta^2-\kappa_{\beta_1}^2)\exp\{-H\lambda_{\alpha_1}\}]\}, \qquad (15\cdot15)
\end{aligned}$$

and S, T, W, U, V, Y have been defined in (14·9). We refer only to conditions in the surface-layer.

The displacements corresponding to the unit P-pulse are then

$$U = \pi \sum_n \int_{-\infty}^{\infty} [\pm \bar{u}_n] \exp\{\mathrm{i}\omega t \mp \mathrm{i}\kappa_n(\omega)\, x\} \frac{\mathrm{d}\omega}{\omega} \quad (\mathscr{R}(\omega) \gtrless 0), \tag{15·16}$$

$$W = \pi \mathrm{i} \sum_n \int_{-\infty}^{\infty} [\bar{w}_n] \exp\{\mathrm{i}\omega t \mp \mathrm{i}\kappa_n(\omega)\, x\} \frac{\mathrm{d}\omega}{\omega}$$

$$= \pi \sum_n \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{\omega} [\pm \bar{w}_n] \exp\{\mathrm{i}\omega t \mp \mathrm{i}\kappa_n(\omega)\, x \pm \mathrm{i}\,\tfrac{1}{2}\pi\} \quad (\mathscr{R}(\omega) \gtrless 0), \tag{15·17}$$

where

$$\bar{u}_n = \kappa \bar{\phi}_n - \frac{\partial \bar{\psi}_n}{\partial z}, \tag{15·18}$$

$$\bar{w}_n = \frac{\partial \bar{\phi}_n}{\partial z} - \kappa \bar{\psi}_n. \tag{15·19}$$

Exact evaluation of the integrals (15·16) and (15·17) in terms of known functions is impossible, but they are of the type suitable for the application of the methods of 'stationary phase' or 'steepest descent'. It happens that the line of stationary phase coincides with the real axis, but so long as no singularities are included between the stationary-phase and steepest-descent curves the methods must be equivalent. Jeffreys (1926 *b*) and Pekeris (1948) developed 'stationary phase' for problems of this type and it will suffice to state the essential results.

Suppose we have an integral in which the exponential part oscillates rapidly compared with the remainder of the integrand so that the main contribution must come from regions of stationary exponent. Then

$$\int_{-\infty}^{\infty} Q(\omega) \exp\{\mathrm{i}(\omega t - \kappa(\omega)\, x)\}\, \mathrm{d}\omega \simeq \sum_n Q(\omega_n) \sqrt{\frac{2\pi}{x\,|\,\kappa''(\omega_n)\,|}} \exp\{\mathrm{i}(\omega_n t - \kappa_n x \pm \tfrac{1}{4}\pi)\}, \tag{15·20}$$

where κ'' denotes $\mathrm{d}^2\kappa/\mathrm{d}\omega^2$; the upper or lower sign is taken according as $\kappa''(\omega_n) \lessgtr 0$ and the summation extends over all ω_n satisfying

$$\left[\left(\frac{\mathrm{d}\kappa}{\mathrm{d}\omega}\right)_{\omega=\omega_n}\right]^{-1} = \frac{x}{t}. \tag{15·21}$$

The condition for the validity of (15·20) is expressed by

$$|\,(\tfrac{5}{24}/x)\,(\kappa'''^2/\kappa''^3 - 15\kappa^{\mathrm{iv}}/\kappa''^2)\,| \ll 1. \tag{15·22}$$

Near a zero of κ'', that is, when $x/t \doteqdot C_0$, a stationary value of group velocity, we must use another form of approximation. If positive values ω_0, κ_0 correspond to the stationary group-velocity C_0 and we define

$$\left.\begin{aligned} a &= t - x\kappa_0' = t - x/C_0, \\ b &= -\tfrac{1}{6}x\kappa_0''' > 0, \quad \text{minimum} \\ &\qquad\qquad < 0, \quad \text{maximum} \\ c &= -\tfrac{1}{24}x\kappa_0^{\mathrm{iv}}, \end{aligned}\right\} \tag{15·23}$$

then

$$\int_{-\infty}^{\infty} Q(\omega) \exp\{\mathrm{i}(\omega t - \kappa(\omega)\, x)\}\mathrm{d}\omega \simeq 4Q(\omega_0)\left[T(a,b)\cos(\omega_0 t - \kappa_0 x) + c\,\frac{\partial^2 T}{\partial a\, \partial b}\sin(\omega_0 t - \kappa_0 x)\right], \tag{15·24}$$

where

$$T = \frac{\pi}{3(2b)^{\frac{1}{3}}} E(v), \quad \frac{\partial^2 T}{\partial a\, \partial b} = \frac{\pi}{b^{\frac{5}{3}}} (\tfrac{2}{3})^{\frac{4}{3}} G(v), \tag{15·25}$$

and $$v = 2\,|\,a\,|^{\frac{3}{2}}/3\sqrt{3}\,|\,b\,|^{\frac{1}{2}}, \tag{15·26}$$

$$\left.\begin{aligned} E(v) &= v^{\frac{1}{3}}[J_{-\frac{1}{3}}(v)+J_{\frac{1}{3}}(v)], \quad t<x/C_{0,\,\text{min.}},\ t>x/C_{0,\,\text{max.}},\\ E(v) &= v^{\frac{1}{3}}[I_{-\frac{1}{3}}(v)-I_{\frac{1}{3}}(v)], \quad t>x/C_{0,\,\text{min.}},\ t<x/C_{0,\,\text{max.}},\end{aligned}\right\} \tag{15·27}$$

$$\left.\begin{aligned} G(v) &= (3^{\frac{1}{2}}/4)\,\{-\tfrac{2}{3}v^{\frac{2}{3}}[J_{-\frac{2}{3}}(v)-J_{\frac{2}{3}}(v)]+\tfrac{1}{2}v^{\frac{5}{3}}[J_{-\frac{1}{3}}(v)+J_{\frac{1}{3}}(v)]\}, \quad t<x/C_{0,\,\text{min.}},\ t>x/C_{0,\,\text{max.}},\\ G(v) &= (3^{\frac{1}{2}}/4)\,\{-\tfrac{2}{3}v^{\frac{2}{3}}[I_{-\frac{2}{3}}(v)-I_{\frac{2}{3}}(v)]+\tfrac{1}{2}v^{\frac{5}{3}}[I_{-\frac{1}{3}}(v)-I_{\frac{1}{3}}(v)]\}, \quad t>x/C_{0,\,\text{min.}},\ t<x/C_{0,\,\text{max.}}.\end{aligned}\right\} \tag{15·28}$$

Hence $$\int_{-\infty}^{\infty} Q(\omega)\exp\{\mathrm{i}(\omega t-\kappa x)\}\,\mathrm{d}\omega \simeq \frac{4\pi Q(\omega_0)}{3\,|\,2b\,|^{\frac{1}{3}}}\,E(v)\cos(\omega_0 t-\kappa_0 x), \tag{15·29}$$

provided $$2^{\frac{5}{3}}G(v)\,c/3^{\frac{1}{2}}b^{\frac{4}{3}}E(v) \ll 1. \tag{15·30}$$

In arriving at the result (15·29) it has been noted that to every positive ω_0 corresponds $-\omega_0$ with the same stationary value of group-velocity.

Now apply (15·20), (15·21) to the evaluation of (15·16), (15·17), the integrals for U, W, noting that the values of ω for which $x/t = (\mathrm{d}\kappa/\mathrm{d}\omega)^{-1}$ occur in pairs $\pm\omega_n$.

Further, since κ is an odd function of ω, c is even and $(\mathrm{d}C/\mathrm{d}\omega)_{\omega_n} = -(\mathrm{d}C/\mathrm{d}\omega)_{-\omega_n}$. Adding the contributions from $\pm\omega_n$, the displacements are

$$U = \sum_n \tfrac{1}{2}\sqrt{(2\pi/x\,|\,\kappa_n''\,|)}\,[\overline{u}_n(\omega_n)/\omega_n]\cos(\omega_n t-\kappa_n x-\tfrac{1}{4}\pi) \qquad (\kappa_0''>0), \tag{15·31}$$

$$= \sum_n \tfrac{1}{2}\sqrt{(2\pi/x\,|\,\kappa_n''\,|)}\,[\overline{u}_n(\omega_n)/\omega_n]\sin(\omega_n t-\kappa_n x-\tfrac{1}{4}\pi) \qquad (\kappa_0''<0), \tag{15·32}$$

$$W = \sum_n \tfrac{1}{2}\sqrt{(2\pi/x\,|\,\kappa_n''\,|)}\,[\overline{w}_n(\omega_n)/\omega_n]\cos(\omega_n t-\kappa_n x+\tfrac{1}{4}\pi) \qquad (\kappa_0''>0), \tag{15·33}$$

$$= \sum_n \tfrac{1}{2}\sqrt{(2\pi/x\,|\,\kappa_n''\,|)}\,[\overline{w}_n(\omega_n)/\omega_n]\sin(\omega_n t-\kappa_n x+\tfrac{1}{4}\pi) \qquad (\kappa_0''<0), \tag{15·34}$$

where contributions from more than one positive value of ω_n are to be superposed. These approximations are valid so long as (15·22) holds. Since x is assumed reasonably large, we need only amend them near points of stationary group-velocity C_0 ($\kappa_0'' = 0$), that is, for times near $t_0 = x/C_0$. Then we use (15·24) and (15·26) yielding

$$U = \frac{2\pi}{3^{\frac{2}{3}}}\left|\frac{x}{H}\frac{1}{\overline{C}^2}\frac{\mathrm{d}^2\overline{C}}{\mathrm{d}\overline{\omega}^2}\right|_0^{-\frac{1}{3}}\frac{\overline{u}_n(\omega_0)}{\overline{\omega}_0}\,E(v)\sin(\omega_0 t-\kappa_0 x), \tag{15·35}$$

$$W = \frac{2\pi}{3^{\frac{2}{3}}}\left|\frac{x}{H}\frac{1}{\overline{C}^2}\frac{\mathrm{d}^2\overline{C}}{\mathrm{d}\overline{\omega}^2}\right|_0^{-\frac{1}{3}}\frac{\overline{w}_n(\omega_0)}{\overline{\omega}_0}\,E(v)\cos(\omega_0 t-\kappa_0 x), \tag{15·36}$$

where $\overline{C}$, $\overline{\omega}$ and v are the dimensionless variables

$$\left.\begin{aligned}\overline{C} &= C/\beta_1,\\ \overline{\omega} &= \kappa Hc/\beta_1,\end{aligned}\right\} \tag{15·37}$$

$$v = \frac{2\sqrt{2}}{3}\left|\frac{1}{\overline{C}^2}\frac{\mathrm{d}^2\overline{C}}{\mathrm{d}\overline{\omega}^2}\right|_0^{\frac{1}{2}}\left(\frac{x}{H}\right)\left|\frac{t-t_0}{t_{\beta_1}}\right|^{\frac{3}{2}}, \quad t_{\beta_1} = x/\beta_1 \tag{15·38}$$

and provided $$\left(\frac{x}{H}\right)^{-\frac{1}{3}}\left\{\frac{1}{\overline{C}^2}\frac{\mathrm{d}^3\overline{C}}{\mathrm{d}\overline{\omega}^3}\Big/\left|\frac{1}{\overline{C}^2}\frac{\mathrm{d}^2\overline{C}}{\mathrm{d}\overline{\omega}^2}\right|^{\frac{4}{3}}\right\}_0\frac{G(v)}{E(v)} \ll 1. \tag{15·39}$$

Pekeris (1948), dealing with two liquid layers, called the disturbance associated with a stationary group-velocity the 'Airy phase'. The term will be used here. The motion is characterized by its regular period, namely, that corresponding to the stationary group-velocity.

Graphs of $E(v)$ and $G(v)/E(v)$ in the neighbourhood of $v = 0$ are reproduced from Pekeris in figure 28; the first determines the envelope of the Airy phase apart from a factor constant for a given x, the second the range of validity of the approximations (15·34) and (15·35).

The striking distinction between the Airy phase and the disturbance which leads up to it lies in the differing rates of attenuation with respect to x. The Airy phase diminishes with distance as $x^{-\frac{1}{3}}$ and the rest as $x^{-\frac{1}{2}}$, so that although the Airy phase is certainly decreasing in intensity it becomes more and more the dominant feature.

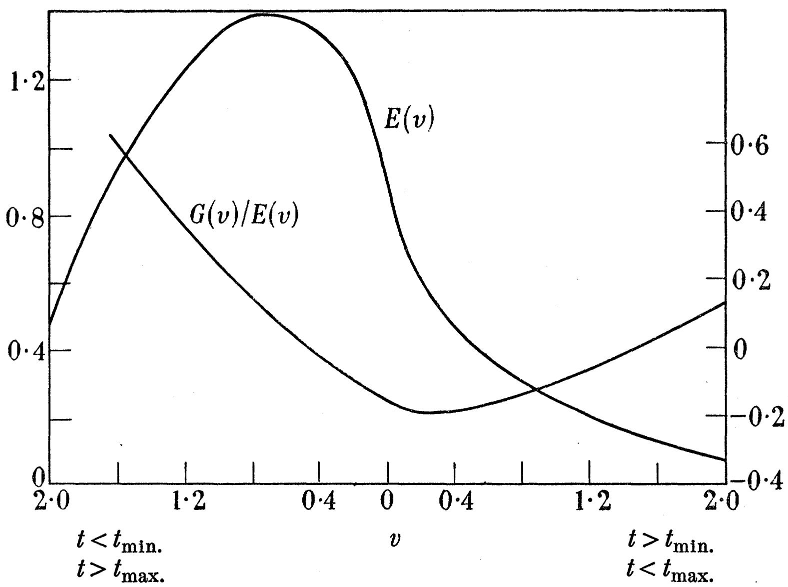

FIGURE 28. Envelope of the Airy phase (from Pekeris).

We shall discuss (15·35) to (15·39) (the expressions for the displacements and the governing condition) with reference to the minimum of the first mode, since it is a feature characteristic of all modes and, it seems, of the group-velocity curves derived in any problem of this type. Provided (15·39) holds, the expressions for U, W may be used for $t<t_0$ and $t>t_0$. When $t<t_0$ there is the alternative approximation (15·20), and as the range x is increased and (as we shall see) the range of validity of the 'Airy' approximation closes in on t_0, so this approximation is valid closer and closer to t_0. When $t>t_0$ there is no alternative and we must consider the importance of the $G(v)$ term, although the failure of the Kelvin approximation is a fair indication of the relative insignificance of the after effect.

First, it may be remarked that the actual peak in the amplitude occurs at an instant preceding t_0 and given by $v \doteqdot 0{\cdot}68$. The 'relative time' $|(t-t_0)/t_{\beta_1}|$ is related to v through (15·38), and we see that

$$|(t-t_0)/t_{\beta_1}| \propto (x/H)^{-\frac{2}{3}}. \tag{15·39}$$

Such an interval corresponds to n complete oscillations (period $2\pi/\kappa_0 C_0$), where

$$n \equiv (t-t_0)\,\kappa_0 C_0/2\pi \propto (x/H)^{\frac{1}{3}}. \tag{15·40}$$

Using

$$\frac{1}{\overline{C}^2}\frac{d^2\overline{C}}{d\overline{\omega}^2} = -0{\cdot}52,$$

$$\frac{1}{\overline{C}^2}\frac{d^3\overline{C}}{d\overline{\omega}^3} = 0{\cdot}28,$$

as obtained for the first mode, we find that

(1) for $(x/H) = 100$: $|(t-t_0)/t_{\beta_1}| \doteqdot 0{\cdot}057$, $n \doteqdot 2{\cdot}7$ oscillations,
(2) for $(x/H) = 1000$: $|(t-t_0)/t_{\beta_1}| \doteqdot 0{\cdot}012$, $n \doteqdot 5{\cdot}8$ oscillations.

Now consider $t > t_0$. From the relations (Watson 1922, pp. 78 and 202)

$$I_{-\nu} - I_\nu = 2(\sin \nu\pi/\pi)\, K_\nu, \tag{15·41}$$

$$K_\nu \sim (z)\ (\pi/2z)^{\frac{1}{2}}\, e^{-z}\,[1 + (4\nu^2 - 1^2)/1!\,8z + \ldots], \tag{15·42}$$

we have that for v sufficiently large ($v \geqslant 4$ is adequate) an approximation to $G(v)/E(v)$ is

$$G(v)/E(v) \doteqdot 3^{\frac{1}{3}} v^{\frac{1}{3}} (v - \tfrac{4}{3})/8. \tag{15·43}$$

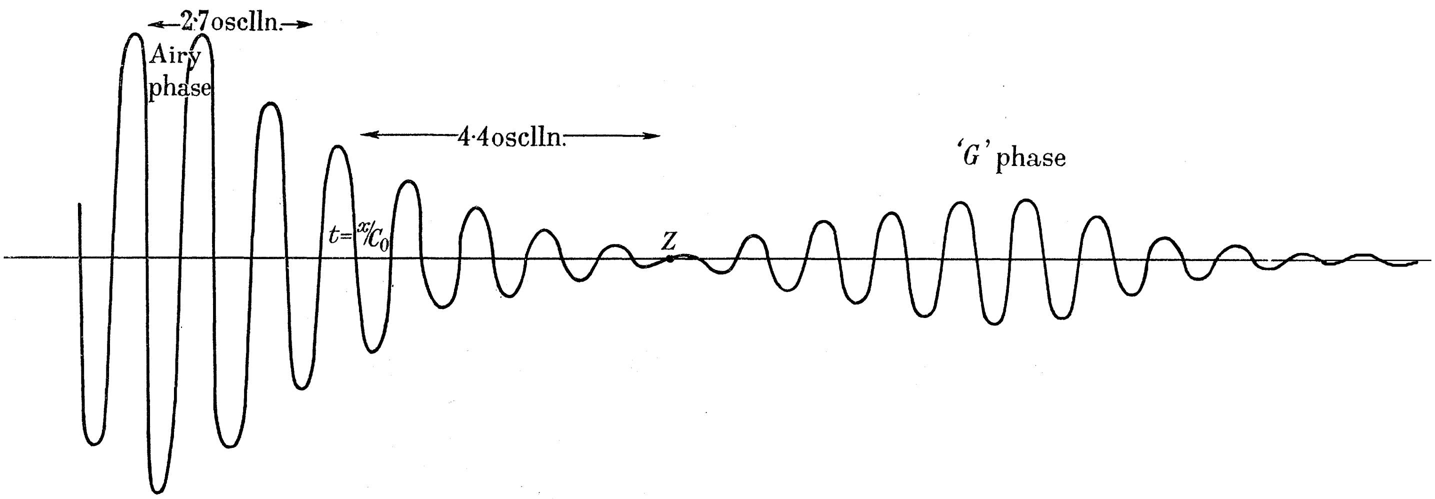

FIGURE 29. Displacement near the Airy phase: $x/H = 100$.

Thus if (15·39) ceases to be valid when the left-hand side of (15·43) is θ, say, then this determines a v proportional to $(x/H)^{\frac{1}{4}}$ (neglecting the $\frac{4}{3}$ compared with v) and from (15·38) we see that this corresponds to an instant t such that

$$|(t-t_0)/t_{\beta_1}| \propto (x/H)^{-\frac{1}{2}}. \tag{15·44}$$

The actual time interval $(t-t_0)$, measured by the number of complete oscillations, is, however, proportional to $(x/H)^{\frac{1}{2}}$.

When neglect of the $G(v)$ term is no longer valid the left-hand side of (15·43) gives just the ratio of the $G(v)$ and $E(v)$ terms, and, since $-\dfrac{1}{\overline{C}^2}\dfrac{d^3\overline{C}}{d\overline{\omega}^3}$ is positive in our problem, there must occur a value of v, that is, an instant at which the resultant amplitude is zero. This is found to occur

(1) for $x/H = 100$: when $v \doteqdot 6{\cdot}25$, $(t-t_0)/t_{\beta_1} \doteqdot 0{\cdot}107$,

or approximately 4·4 oscillations after t_0.

(2) for $x/H = 1000$: when $v \doteqdot 10{\cdot}2$, $(t-t_0)/t_{\beta_1} \doteqdot 0{\cdot}032$,

or approximately 13·2 oscillations after t_0.

The amplitude should then increase slightly to a maximum then fall off exponentially governed by $G(v)$. Figure 29 shows the theoretical trace of the Airy phase for the case

$x/H = 100$; the vertical scale is exaggerated to the right for convenience. The 'G' phase is introduced. It may be verified at this stage that neglect of higher-order terms in the original integrals is still justified.

If we take $\frac{1}{100}$th of the maximum amplitude to be an estimate of the least perceptible displacement, then before the zero point Z is reached the modulus of both the $G(v)$ and $E(v)$ parts for $x/H = 100$ will be well below the limit of perceptibility. But it is seen from the equation (15·39) that the value of v determined is very sensitive to changes in $\frac{1}{\overline{C}^2}\frac{d^2\overline{C}}{d\overline{\omega}^2}$ and $\frac{1}{\overline{C}^2}\frac{d^3\overline{C}}{d\overline{\omega}^3}$ as well as in x. A change by a factor of two only might bring the 'G' phase within the recordable range even at the distance $x = 100H$. This feature should not be confused with a secondary Airy phase due to a multiple explosion. If magnitude alone does not distinguish the two there is the time of arrival. An 'after-shock' makes itself felt at a fixed time after the main shock; the feature under discussion, or more exactly the zero amplitude which precedes it, occurs at a time after t_0 proportional to $(x/H)^{\frac{1}{3}}$.

Puzzling features of seismograms are the long trains of regular waves—regular in period and amplitude—which are frequently found to follow the Airy phase. The discussion above shows fairly conclusively that the explanation is to be sought in some external mechanism of which this theory does not take account. Suggestions that they represent the free vibrations of the ground on which the observation point is situated relative to the main earth mass invite some justification.

The equations (15·31) to (15·36) were used to calculate the displacements contributed by successive modes; period-velocity relations, including values of $\kappa_0', \ldots, \kappa_0^{\text{iv}}$ for all ω and in the first five modes of vibration were read from an extended table 1 and figure 26. In the first and second modes, which each show a single stationary value of group-velocity (a minimum), U and W were determined completely for sources located in the layer close to the free surface, at a depth $\frac{1}{2}H$, and near the interface, and for observation points at horizontal range x of $100H$ and $1000H$.

Owing to the form of the expressions for U and W, the determining influence of negative exponentials like $\exp\{-h\lambda_{\alpha_1}\}$, and the steady decrease of associated period, the contributions from successive modes rapidly diminish; but the earliest arrival from the first mode is at $t = x/0{\cdot}9194\ldots\beta_2$, whereas from the second and higher modes it is at $t = x/\beta_2$, so that at least until shortly after $t = x/0{\cdot}9194\ldots\beta_2$ the second mode contribution should be significant. Moreover, as the respective values of minimum group-velocity are quite distinct, that of the second mode exceeding that of the first, the Airy phase of the second mode may well be distinguishable as a prominence preceding the Airy phase of the first mode. The condition would be that x be sufficiently great for the additional power of $x^{\frac{1}{6}}$ associated with the Airy phase to more than counterbalance the general amplitude decrease with n referred to above.

Figures 30 to 34 shows the displacements as functions of period and against time; one unit on the vertical scale represents a convenient dimensionless multiple of (L^2/H), where L is the unit of length and H the layer depth in terms of these length units.

For a surface explosion and surface observation point, i.e. putting $h = 0$, $z = 0$, the calculations recorded an infinite displacement corresponding to the very short period waves arriving at $t = x/0{\cdot}9194\ldots\beta_1$ in the first mode and at $t = x/\beta_1$ in the second and higher.

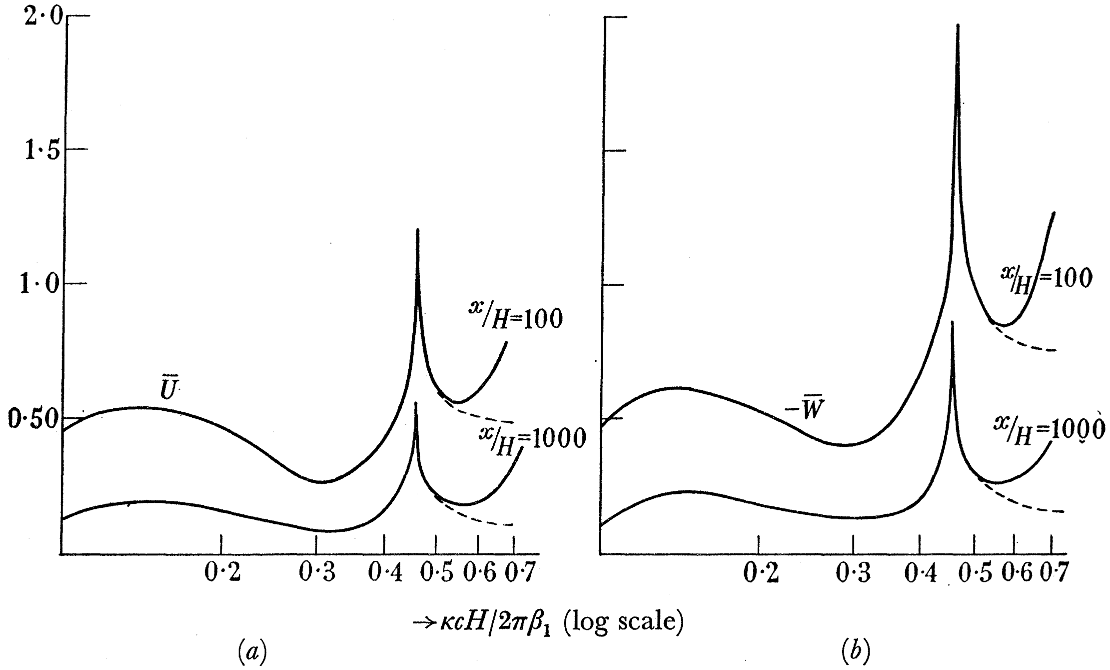

FIGURE 30. Displacements as functions of 1/period: first mode, $h=0$, $z=0$, $\beta_2/\beta_1=\alpha_2/\alpha_1=4/3$, (*a*) horizontal, (*b*) vertical (broken line represents modification for h small but non-zero).

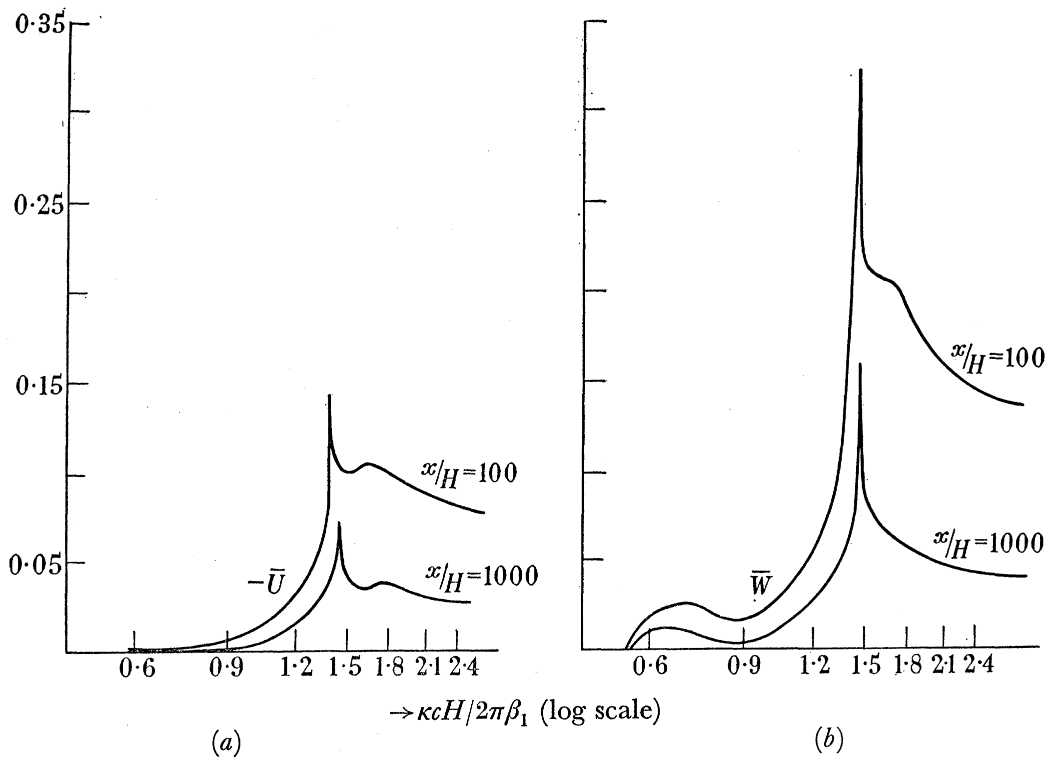

FIGURE 31. Displacements as functions of 1/period: second mode: $h=0$, $z=0$, $\beta_2/\beta_1=\alpha_2/\alpha_1=4/3$, (*a*) horizontal, (*b*) vertical.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

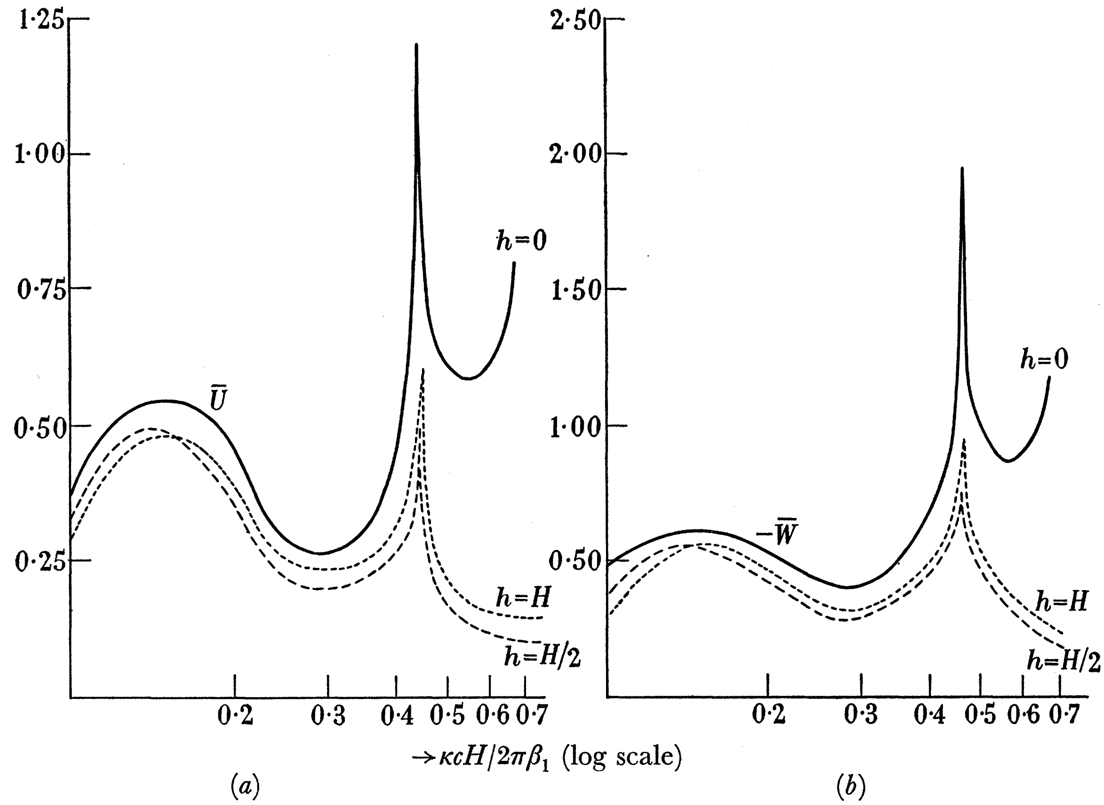

FIGURE 32. Displacements for different depths of source: first mode, $x/H = 100$, $z = 0$, $\beta_2/\beta_1 = \alpha_2/\alpha_1 = 4/3$, (a) horizontal, (b) vertical.

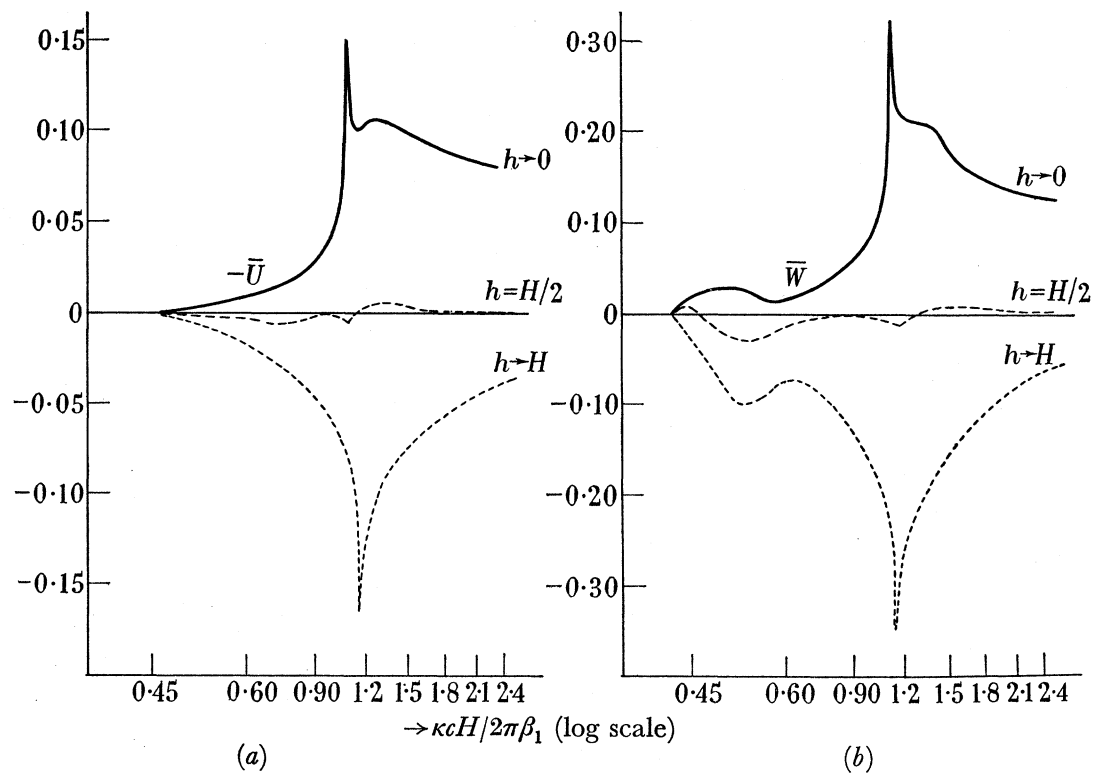

FIGURE 33. Displacements for different depths of source: second mode, $x/H = 100$, $z = 0$, $\beta_2/\beta_1 = \alpha_2/\alpha_1 = 4/3$, (a) horizontal, (b) vertical.

This suggests the failure of the solution to represent the actual conditions. Press *et al.* (1948) encountered a similar difficulty in treating the liquid-solid case.

On a physical argument, the case of an explosion about a line in the surface must present a vastly different problem from that of one occurring at some depth; in the latter we are justified in assuming an initially symmetrical, cylindrical spread of energy appropriate to the elastic medium I, and, implicit in our theory, no permanent change of form of the surface. When the axis of the explosion is in the surface no such simple spread of the energy could take place since much of it is lost to the free space above.

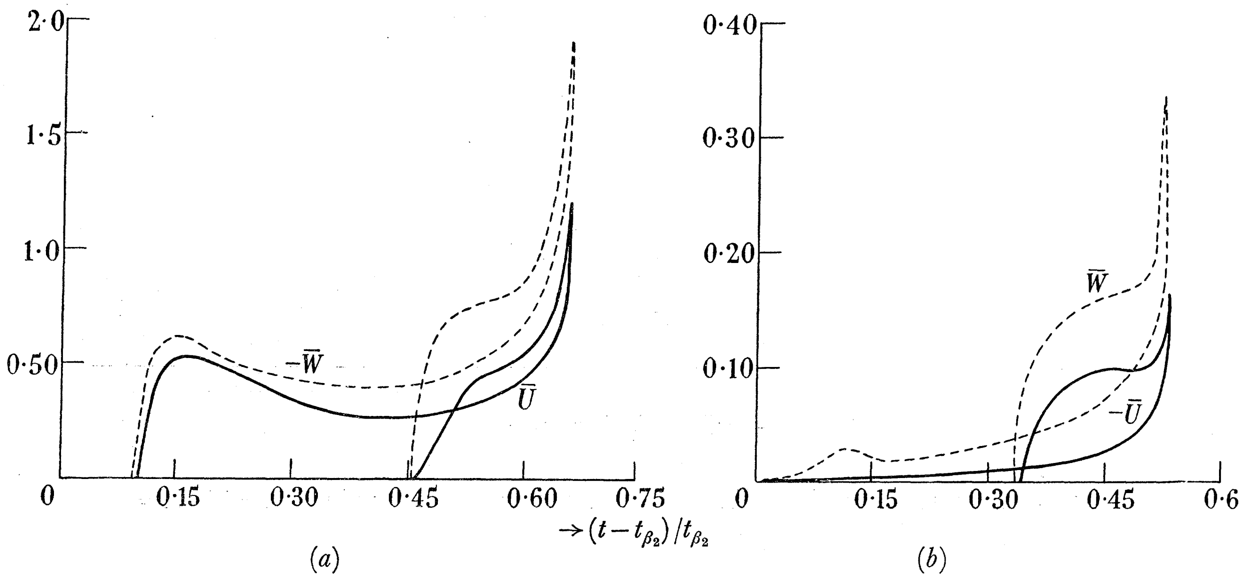

FIGURE 34. Displacements as functions of 1/period: first mode, $h=0$, $z=0$, $\beta_2/\beta_1=\alpha_2/\alpha_1=4/3$, (*a*) horizontal, (*b*) vertical.

Our method of solution was to let Φ_0 represent the initial cylindrical explosion of the pulse in medium I, and to this we added $-\Phi_r$, equivalent to an equal and opposite pulse at the image position; in this case they are superposed and the combined effect is nil. We proceeded to add Φ and Ψ to satisfy the boundary condition of zero stress at the surface, which for a surface origin must be contradictory. Despite this, if the mathematical solution for $h=0$ gave the limit of the disturbance as h tended to zero then it would have physical meaning and would, in so far as we gave it this interpretation, be valid. We shall see that this is not so for κH very great but it is not difficult to estimate the deviation and the effect of a positive depth h however small. It is Φ and Ψ which provide the normal mode motion. The amplitudes are determined through the terms in the boundary conditions contributed by Ψ_{0r} and given by a column vector (see p. 219) which for convenience we shall write as

$$(-\kappa \sinh(h\lambda_{\alpha_1}) \exp\{-H\lambda_{\alpha_1}\}/\lambda_{\alpha_1};\ -\sinh(h\lambda_{\alpha_1}) \exp\{-H\lambda_{\alpha_1}\};\ 2\kappa \sinh(h\lambda_{\alpha_1}) \exp\{-H\lambda_{\alpha_1}\};\ (2\kappa^2-\kappa^2_{\beta_1}) \exp\{-H\lambda_{\alpha_1}\} \sinh(h\lambda_{\alpha_1})/\lambda_{\alpha_1};\ -2\kappa \exp\{-h\lambda_{\alpha_1}\};\ 0).$$

Since these terms are derived from Φ_{0r} and its derivatives, and we saw that for $h=0$ Φ_{0r} is zero, these should all be zero; but the boundary conditions have been so formulated that putting $h=0$ in the above we obtain

$$(0,\ 0,\ 0,\ 0,\ -2\kappa,\ 0) \quad \text{for all } \kappa H.$$

On the other hand if $h>0$ but very small the value changes from $(0, 0, 0, 0, -2\kappa, 0)$ for long periods to $(0, 0, 0, 0, 0, 0)$ for infinitely short periods, so that it is not surprising that putting $h = 0$ we derive, wrongly, infinite displacements for the very short periods.

Closer examination of the U, W expressions shows that $h = 0$ gives the limit as $h \to 0$ unless κH is very great, and then the substitution '$h>0$, but very small' ensures the vanishing of the displacements for very short periods. The dotted line in figure 30 shows the effect of a small positive h on the U, W calculations for $h = 0$.

It might appear that $h = H$ defines another case where it is not physically possible to have the initial symmetrical energy-spread in medium I as assumed in the theory, for the pulse would here be right on the interface; but we can show that for all periods the solution represents the limit as $h \to H$ from within the surface layer and so has a physical meaning and, thus interpreted, is valid throughout.

To check some of these arguments quantitatively we must study the limiting forms of the expressions for U, W as $\kappa H \to \infty$. This can be done through the following considerations.

In the process of calculating U, W numerically we must evaluate $(\partial\Delta/\partial\kappa)_\omega$. As a function of κ and ω, Δ' contains these variables in coefficients and exponents; as a function of κ and c it contains κ only as factor κ^{10} and in the exponentials $\exp\{-\kappa H\sqrt{(1-c^2/\alpha_1^2)}\}$, etc., and it is easy to write down the expression for $(\partial\Delta/\partial\kappa)_c$. For $c<\beta_1$ it is given by

$$\begin{aligned}\left(\frac{\partial\Delta}{\partial\kappa}\right)_c = (H/\kappa)\{&[-4\kappa^2\lambda_{\alpha_1}\lambda_{\beta_1}S-(2\kappa^2-\kappa_{\beta_1}^2)^2\,T]\,[\lambda_{\alpha_1}\exp\{H\lambda_{\alpha_1}\}\cosh H\lambda_{\beta_1}+\lambda_{\beta_1}\exp\{H\lambda_{\alpha_1}\}\sinh H\lambda_{\beta_1}]\\ &+[4\kappa^2\lambda_{\alpha_1}\lambda_{\beta_1}T+(2\kappa^2-\kappa_{\beta_1}^2)^2S]\,[\lambda_{\alpha_1}\exp\{H\lambda_{\alpha_1}\}\sinh H\lambda_{\beta_1}+\lambda_{\beta_1}\exp\{H\lambda_{\alpha_1}\}\cosh H\lambda_{\beta_1}]\\ &+[-4\kappa^2\lambda_{\alpha_1}\lambda_{\beta_1}W+(2\kappa^2-\kappa_{\beta_1}^2)^2\,U]\,[-\lambda_{\alpha_1}\exp\{-H\lambda_{\alpha_1}\}\cosh H\lambda_{\beta_1}+\lambda_{\beta_1}\exp\{-H\lambda_{\alpha_1}\}\sinh H\lambda_{\beta_1}]\\ &+[4\kappa^2\lambda_{\alpha_1}\lambda_{\beta_1}U-(2\kappa^2-\kappa_{\beta_1}^2)^2\,W]\,[-\lambda_{\alpha_1}\exp\{-H\lambda_{\alpha_1}\}\sinh H\lambda_{\beta_1}+\lambda_{\beta_1}\exp\{-H\lambda_{\alpha_1}\}\cosh H\lambda_{\beta_1}]\}.\end{aligned} \tag{15·45}$$

Now
$$\left(\frac{\partial\Delta}{\partial\kappa}\right)_\omega = \left(\frac{\partial\Delta}{\partial\kappa}\right)_c + \left(\frac{\partial\Delta}{\partial c}\right)_\kappa \left(\frac{\partial c}{\partial\kappa}\right)_\omega, \tag{15·46}$$

and we have tabulated c against κ satisfying $\Delta = 0$, so that along the κ-c curve of any mode

$$\delta\Delta = 0 = \left(\frac{\partial\Delta}{\partial\kappa}\right)_c \mathrm{d}\kappa + \left(\frac{\partial\Delta}{\partial c}\right)_\kappa \mathrm{d}c. \tag{15·47}$$

Therefore
$$\begin{aligned}\left(\frac{\partial\Delta}{\partial\kappa}\right)_\omega &= \left(\frac{\partial\Delta}{\partial\kappa}\right)_c \left[1-\left(\frac{\partial c}{\partial\kappa}\right)_\omega \left(\frac{\mathrm{d}\kappa}{\mathrm{d}c}\right)_{\substack{\text{along}\\ \text{curve }\Delta=0}}\right]\\ &= -\left(\frac{\partial\Delta}{\partial\kappa}\right)_c \left(\frac{C}{c-C}\right)_{\Delta=0}.\end{aligned} \tag{15·48}$$

We shall now confine attention to the first mode in which $\kappa H \to \infty$ as $c \to 0{\cdot}9194\ldots\beta_1$, the solution of
$$(2-c^2/\beta_1^2)^2-4(1-c^2/\beta_1^2)^{\frac{1}{2}}(1-c^2/\alpha_1^2)^{\frac{1}{2}} = 0.$$

In (14·17) the limiting form of $\kappa(\mathrm{d}c/\mathrm{d}\kappa)$ for κH large is given, and it is seen that it tends to zero like
$$\kappa H\exp\{-2\kappa H(1-0{\cdot}9194\ldots^2)^{\frac{1}{2}}\}. \tag{15·49}$$

Thus C may be replaced by the limiting phase-velocity. From (15·45) it is seen that the important terms of $(\partial\Delta/\partial\kappa)_c$ are

$$\begin{aligned}\tfrac{1}{2}\exp\{H\lambda_{\alpha_1}\}(H/\kappa)\,&[\exp\{H\lambda_{\beta_1}\}(S-T)(\lambda_{\alpha_1}+\lambda_{\beta_1})\{(2\kappa^2-\kappa_{\beta_1}^2)^2-4\kappa^2(1-c^2/\alpha_1^2)^{\frac{1}{2}}(1-c^2/\beta_1^2)^{\frac{1}{2}}\}\\ &+\exp\{-H\lambda_{\beta_1}\}(S+T)(\lambda_{\alpha_1}-\lambda_{\beta_1})\{(2\kappa^2-\kappa_{\beta_1}^2)^2+4\kappa^2(1-c^2/\alpha_1^2)^{\frac{1}{2}}(1-c^2/\beta_1^2)^{\frac{1}{2}}\}],\end{aligned}$$

and since $$(2\kappa^2-\kappa_{\beta_1}^2)^2-4\kappa^2(1-c^2/\alpha_1^2)^{\frac{1}{2}}(1-c^2/\beta_1^2)^{\frac{1}{2}}\to 0 \quad \text{like } \exp\{-2H\lambda_{\beta_1}\}, \tag{15·50}$$

so $$\left(\frac{\partial\Delta}{\partial\kappa}\right)_c\to\infty \quad \text{like } \kappa^9\exp\{H(\lambda_{\alpha_1}-\lambda_{\beta_1})\}. \tag{15·51}$$

For a surface observation both $\Delta'\overline{U}$ and $\Delta'\overline{W}$ have predominant terms varying as $\kappa^9\exp\{H(\lambda_{\alpha_1}+\lambda_{\beta_1})-h\lambda_{\alpha_1}\}$; combining this result with (15·49) and (15·51) we see that our theory gives U and W varying, for large κH, as $\kappa H\exp\{-\kappa h(1-0{\cdot}9194\ldots^2)^{\frac{1}{2}}\}$. Thus for any positive h, however small, both U and W tend uniformly to zero as the period becomes infinitesimally short; but for $h=0$ we derive infinite displacements and the solution is invalid.

By the same methods we may show that

$$\lim_{\substack{z=0\\ h\to 0\\ \kappa H\to\infty\\ c\to 0{\cdot}9194\ldots\beta_1}}\left(\frac{W}{U}\right)=\frac{(2-c^2/\beta_1^2)^2-2(1-c^2/\alpha_1^2)^{\frac{1}{2}}(1-c^2/\beta_1^2)^{\frac{1}{2}}}{(1-c^2/\beta_1^2)^{\frac{1}{2}}(2-c^2/\beta_1^2)-2(1-c^2/\beta_1^2)^{\frac{1}{2}}},$$
$$=-1{\cdot}468\ldots, \tag{15·52}$$

the value obtained for Rayleigh waves in a uniform semi-infinite medium of the material of the surface layer.

Similar arguments may be applied to the second and higher mode solutions for which $c\to\beta_1$ $(\bar\lambda_{\beta_1}\to 0)$ as $\kappa H\to\infty$. T, U, V all tend to zero as $c\to\beta_1$ but $T/\bar\lambda_{\beta_1}$, $U/\bar\lambda_{\beta_1}$ and $V/\bar\lambda_{\beta_1}$ are finite and non-zero.

The wave-equation takes the limiting form

$$\tan H\bar\lambda_{\beta_1}=\frac{-\bar\lambda_{\beta_1}[(2-c^2/\beta_1^2)^2\overline{T}/\bar\lambda_{\beta_1}-4(1-c^2/\alpha_1^2)^{\frac{1}{2}}(c^2/\beta_1^2-1)^{\frac{1}{2}}\overline{S}]}{(2-c^2/\beta_1^2)^2\overline{S}} \tag{15·53}$$

and the right-hand side approaches the value $-3{\cdot}256\ldots\bar\lambda_{\beta_1}$ as $c\to\beta_1$. Further,

$$\lim_{\substack{z=0\\ h\to 0\\ \kappa H\to\infty\\ \bar\lambda_{\beta_1}\to 0}}\left(\frac{W}{U}\right)=\frac{[2\lambda_{\alpha_1}\bar\lambda_{\beta_1}\overline{S}-(2\kappa^2-\kappa_{\beta_1}^2)\,\overline{T}]+[(2\kappa^2-\kappa_{\beta_1}^2)\overline{S}+2\lambda_{\alpha_1}\bar\lambda_{\beta_1}\overline{T}]\tan H\bar\lambda_{\beta_1}}{\bar\lambda_{\beta_1}(S-T\tan H\bar\lambda_{\beta_1})} \tag{15·54}$$

and here the right-hand side approaches the value 1·633 ....

We have assumed in this treatment that the initial pulse is of P-type. Whereas this is probable true of experimentally produced explosions it is well known that natural earthquakes are rich in distortional waves. We saw, however, that the principal features of the surface-wave pattern are characteristic of the system and independent of the generating explosion—the arrival times of the long and short waves in the first mode, the cut-off velocities and periods in the higher modes, the location of the Airy phase and its eventual predominance. The corresponding analysis for an initial S-pulse may therefore be expected to reveal only minor differences in the form of the displacements.

16. The generation of Love waves

We have so far considered only a line source of P-waves and what are actually SV-waves with cylindrical symmetry.

The disturbance due to a spherically symmetrical source of SH-waves in a surface layer was considered by Jeffreys (1931) using the expansion method of Sommerfeld. He identified successive pulses with the reflected and refracted paths of the ray theory and attempted to recombine certain of these to reproduce the dispersive train of Love waves, in whose generation he was primarily interested.

For completeness the corresponding analysis for a line source of *SH*-waves will be briefly outlined. With the elastic system of figure 1 we wish to consider the motion due to an initial uniform disturbance along the line $x = 0$, $z = h$. This may be written $(0, V_0, 0)$, where we suppose that V_0 varies in time like a unit Heaviside function.

The equations of motion reduce to the single equation

$$\mu\nabla^2 v = \rho\frac{\partial^2 v}{\partial t^2}, \tag{16·1}$$

of which the appropriate solution, varying in time as $e^{i\omega t}$, is

$$v_0 = \pi\kappa_{\beta_1}\mathrm{H}_{i_0}(\kappa_{\beta_1}\varpi)\,e^{i\omega t}, \tag{16·2}$$

which may be written (§ 2)

$$v_0 = 2i\kappa_{\beta_1}\int_0^\infty \exp\{-(h-z)\,\lambda_{\beta_1}\}\cos\zeta x\frac{d\zeta}{\lambda_{\beta_1}}\,e^{i\omega t} \quad (0\leqslant z\leqslant h)$$

$$= 2i\kappa_{\beta_1}\int_0^\infty \exp\{-(z-h)\,\lambda_{\beta_1}\}\cos\zeta x\frac{d\zeta}{\lambda_{\beta_1}}\,e^{i\omega t} \quad (h\leqslant z\leqslant H), \tag{16·3}$$

and V_0 is derived from v_0 through the relation

$$V_0 = \frac{1}{2\pi i}\int_\Omega v_0\frac{d\omega}{\omega}. \tag{16·4}$$

The constant π in (16·2) is introduced for algebraic convenience and the factor κ_{β_1} to bring the dimensions of v_0 in line with those of u_0, w_0 of the earlier work.

Adding an equal source at the image line, $x = 0$, $z = -h$, we ensure the vanishing of the only surface stress which is not identically zero; satisfying the continuity conditions at the interface we finally find that the disturbance is given by the sum of displacements

$$v_{0r} = v_0+v_r = 4i\kappa_{\beta_1}\int_0^\infty \exp\{-h\lambda_{\beta_1}\}\cosh(z\lambda_{\beta_1})\cos\zeta x\frac{d\zeta}{\lambda_{\beta_1}}e^{i\omega t} \quad (0\leqslant z\leqslant h) \tag{16·5}$$

$$= 4i\kappa_{\beta_1}\int_0^\infty \exp\{-z\lambda_{\beta_1}\}\cosh(h\lambda_{\beta_1})\cos\zeta x\frac{d\zeta}{\lambda_{\beta_1}}e^{i\omega t} \quad (h\leqslant z\leqslant H), \tag{16·6}$$

and

$$v = 8i\kappa_{\beta_1}\int_0^\infty \cosh(z\lambda_{\beta_1})\cosh(h\lambda_{\beta_1})$$
$$\times\exp\{-2H\lambda_{\beta_1}\}\left[\frac{\mu_1\lambda_{\beta_1}+\mu_2\lambda_{\beta_2}}{\mu_1\lambda_{\beta_1}-\mu_2\lambda_{\beta_2}}-\exp\{-2H\lambda_{\beta_1}\}\right]^{-1}\cos\zeta x\frac{d\zeta}{\lambda_{\beta_1}}e^{i\omega t}. \tag{16·7}$$

Expanding (16·7) in negative powers of the exponential we have

$$v = 2i\kappa_{\beta_1}\sum_{n=1}^\infty\int_0^\infty[\exp\{(h+z)\,\lambda_{\beta_1}\}+\exp\{-(h-z)\,\lambda_{\beta_1}\}+\exp\{-(z-h)\,\lambda_{\beta_1}\}+\exp\{-(h+z)\,\lambda_{\beta_1}\}]$$
$$\times q^n\exp\{-2nH\lambda_{\beta_1}\}\cos\zeta x\frac{d\zeta}{\lambda_{\beta_1}}e^{i\omega t}, \tag{16·8}$$

$$= \sum_{n=1}^\infty v_n, \quad \text{say,}$$

where

$$q = (\mu_1\lambda_{\beta_1}-\mu_2\lambda_{\beta_2})/(\mu_1\lambda_{\beta_1}+\mu_2\lambda_{\beta_2}), \tag{16·9}$$

and the signs of the radicals are chosen so that $\mathscr{R}(\lambda_{\beta_2})>0$, $\mathscr{R}(\lambda_{\beta_1})>0$.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

We now replace the real axis of ζ by the Sommerfeld contour, which consists of only two loops surrounding the cuts $\mathscr{R}(\lambda_{\beta_2}) = 0$, $\mathscr{R}(\lambda_{\beta_1}) = 0$ (figure 35); the radicals λ_{α_2}, λ_{α_1} do not arise in this problem, and it is readily shown that the denominator in (16·7) has no zeros which do not lie indefinitely close to the cuts.

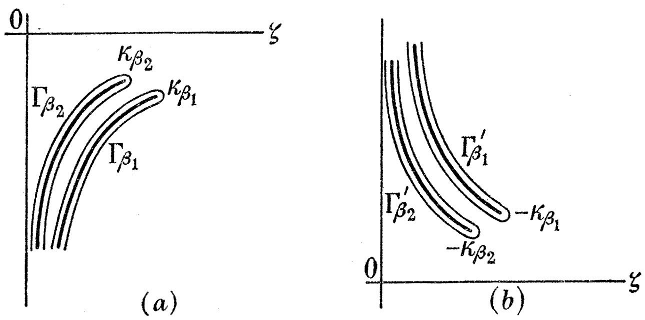

FIGURE 35. Sommerfeld contour for initial *SH*-pulse, (*a*) $\mathscr{R}(\omega) > 0$, (*b*) $\mathscr{R}(\omega) < 0$.

Evaluating the integrals approximately by the methods of § 9 we find that the contributions from Γ_{β_1} may be written

$$\left.\begin{aligned} v_{0,\beta_1} &= -\sqrt{\left(\frac{2\pi}{\mathrm{i}^3 x\beta_1}\right)}\,\omega^{\frac{1}{2}} \exp\{i\omega[t - x/\beta_1 - (h-z)^2/2x\beta_1]\} \quad (0 \leqslant z \leqslant h), \\ V_{0,\beta_1} &= \sqrt{\left(\frac{2}{x\beta_1\tau}\right)}\,H(\tau), \quad \text{where} \quad \tau = t - x/\beta_1 - (h-z)^2/2x\beta_1, \end{aligned}\right\} \tag{16·10}$$

$$\left.\begin{aligned} v_{r,\beta_1} &= -\sqrt{\left(\frac{2\pi}{\mathrm{i}^3 x\beta_1}\right)}\,\omega^{\frac{1}{2}} \exp\{\mathrm{i}\omega[t - x/\beta_1 - (h+z)^2/2x\beta_1]\}, \\ V_{r,\beta_1} &= \sqrt{\left(\frac{2}{x\beta_1\tau}\right)}\,H(\tau), \quad \text{where} \quad \tau = t - x/\beta_1 - (h+z)^2/2x\beta_1, \end{aligned}\right\} \tag{16·11}$$

$$\left.\begin{aligned} v_{n,\beta_1} &= (-1)^{n+1}\sqrt{\left(\frac{2\pi}{\mathrm{i}^3 x\beta_1}\right)}\,\omega^{\frac{1}{2}} \sum_{\substack{4\ \text{combns} \\ \text{of sign}}} \exp\{\mathrm{i}\omega[t - x\beta_1 - (2nH \pm h \pm z)^2/2x\beta_1]\}, \\ V_{n,\beta_1} &= (-1)^n\sqrt{\left(\frac{2}{x\beta_1\tau}\right)}\,H(\tau), \quad \text{where} \quad \tau = t - x/\beta_1 - (2nH \pm h \pm z)/2x\beta_1. \end{aligned}\right\} \tag{16·12}$$

We see that V_{r,β_1} and V_{n,β_1} are of the same shape as the initial pulse; each component corresponds to travel with velocity β_1 along a minimum-time path involving $0, 1, 2, \ldots$ reflexions at the interface. The factor $(-1)^n$ represents the change of phase on reflexion at an interface. The approximations depend for their validity on x being sufficiently large compared with h, z, $2nH$ for the main contribution to the integrands to come from the immediate neighbourhood of the branch-point κ_{β_1}, and we have substituted actual values at κ_{β_1} in all but the exponents. We have thus suppressed the factor q^n whose modulus is everywhere less than unity (except at κ_{β_1} and κ_{β_2} where q is -1 and $+1$ respectively) and whose presence represents the decrease of amplitude associated with successive reflexions at the interface. In terms of geometrical theory, we have chosen x so large that the angle of incidence is sufficiently near 90° for the loss of energy to the lower medium to be negligible; but for a given x, as the number of reflexions (given by n) increases, the angle of incidence decreases and the energy loss may not be disregarded.

Neither V_0 nor V_r contains the radical λ_{β_2} so that no contribution is derived from Γ_{β_2}. Close to κ_{β_2} we may write

$$(q)^n = [(\mu_1\lambda_{\beta_1}-\mu_2\lambda_{\beta_2})/(\mu_1\lambda_{\beta_1}+\mu_2\lambda_{\beta_2})]^n \doteqdot 1-2n\mu_2\lambda_{\beta_2}/\mu_1\lambda_{\beta_1} \tag{16·13}$$

and the part containing λ_{β_2} provides the only contribution to V_{n,β_2} from Γ_{β_2}. We find

$$\left.\begin{aligned} v_{n,\beta_2} &= 2\sqrt{\left(\frac{2\pi}{\mathrm{i}}\right)}\frac{n\mu_2\kappa_{\beta_1}}{\kappa_{\beta_2}\,.\,\kappa^2_{\widehat{\beta_1\beta_2}}}(x\beta_2-\overline{2nH\pm h\pm z}\,\widehat{\beta_1\beta_2})^{-\frac{3}{2}}\,\omega^{-\frac{1}{2}} \\ &\qquad\times\exp\{\mathrm{i}\omega[t-x/\beta_2-(2nH\pm h\pm z)]/\widehat{\beta_1\beta_2}\}, \\ V_{n,\beta_2} &= 4\sqrt{2}\,\frac{n\mu_2\kappa_{\beta_1}}{\mu_1\kappa_{\beta_2}\,.\,\kappa^2_{\widehat{\beta_1\beta_2}}}(x\beta_2-\overline{2nH\pm h\pm z}\,\widehat{\beta_1\beta_2})^{-\frac{3}{2}}\,\tau^{\frac{1}{2}}\,H(\tau), \\ &\qquad\text{where}\quad \tau = t-x/\beta_2-\overline{2nH\pm h\pm z}/\widehat{\beta_1\beta_2}. \end{aligned}\right\} \tag{16·14}$$

Each component is a sharp pulse whose travel time corresponds to a minimum-time path in which the disturbance travels most of the way with velocity β_2 in the second medium, being refracted at grazing angle along the interface. The value of n determines the number of preliminary reflexions from the interface in the surface layer (figure 36). The displacements rise steeply but smoothly from zero; the velocity varies like the initial displacement V_0, suddenly acquiring a great value at the onset time and thereafter decreasing.

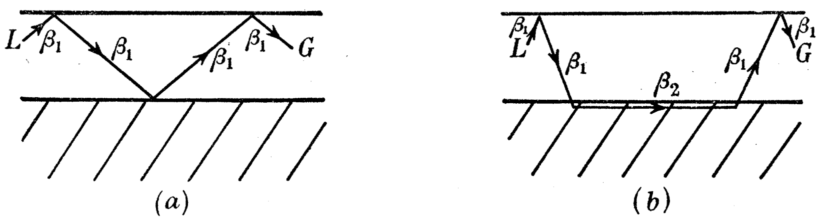

FIGURE 36. Travel paths of the (a) V_{n,β_1}-, (b) V_{n,β_2}-pulses, $n=1$, $h_1=0$, $h_2=2H+n+z$.

We note that these contributions have all the same sign and are therefore cumulative; it was to these that Jeffreys looked for the Love wave phase.

Alternatively the investigation of the surface waves may be approached directly, reverting to real ω and, as in the case of initial P- and SV-pulses, considering only that part of the modified ζ-contour which gives significant contributions at great distances. A complete study of the dispersive Love wave train will be reserved for a future paper.

17. EXTENSION OF THE THEORY TO MULTILAYERED SYSTEMS

It remains to consider how far this theory of a line source in a medium with a single surface layer may be extended to multilayered media.

The pulse representation, we saw, has a natural extension to any number of layers. Pulses should be felt corresponding to travel by every one of the minimum-time paths predicted by the ray theory. Analogous diffraction pulses may be deduced.

At great distances interference between pulses becomes important and the dispersive train of surface waves becomes the predominant feature; if we are to use records of this and in particular of the Airy phases to deduce the nature of the material underlying a given surface, then we must be familiar with the group velocity curves of a variety of media.

The theory of Love waves in double- and triple-surface layers was given by Stoneley (1937); the corresponding analysis for Rayleigh waves leads to a wave equation which is intractable for even two surface layers. In certain cases, and for a limited range of period, it may be possible to substitute for a multilayered system one in which the properties vary continuously and some special examples of continuous variation have been treated by Stoneley (1934), Pekeris (1935), the author (Newlands 1950) and others. The present study would seem to stress the desirability of further investigation into the effects of heterogeneity.

I wish to express my thanks to Dr R. Stoneley, F.R.S., for valuable discussion and constant encouragement during the course of this work; also to the Mathematical Laboratory, Cambridge, for computational facilities and to the Department of Scientific and Industrial Research for a research training grant which made the investigation possible.

Appendix 1. Improved approximations to diffraction terms

It is not a serious drawback that from the loops Γ_{β_1} and Γ_{β_2} we have, in general, been able only to derive approximations to $\dot{U}$ and $\dot{W}$ and not to the horizontal and vertical displacements themselves. Primarily, we are interested in changes in the displacements, since it is these which show up on the seismogram and are of practical importance. On the other hand, the results for ${}_p\dot{U}^{(4)}$, ${}_p\dot{W}^{(4)}$, typical components, are seen to lack the sharpness of

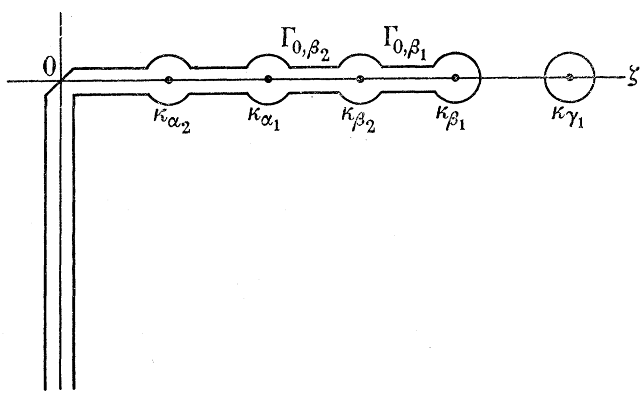

Figure 37. Limiting form of the loop Γ when ω is real.

contributions from Γ_{α_2} and Γ_{α_1}; for this reason it was considered advisable to attempt to check the approximations by some independent method as Lapwood did when treating what were actually the ${}_p\phi^{(1)}$, ${}_p\psi^{(1)}$, ${}_s\phi^{(1)}$, ${}_s\psi^{(1)}$ terms in our problem. Following Lapwood (1949, p. 89) the limiting case of ω real was considered, when the contour Ω approaches Ω_0 (the real axis from $-\infty$ to $+\infty$) and, for $\omega > 0$, the four loops Γ_{α_2}, Γ_{α_1}, Γ_{β_2}, Γ_{β_1} merge into a single loop Γ_0 running from $-i\infty$ round κ_{α_2}, κ_{α_1}, κ_{β_2}, κ_{β_1} on the real axis (figure 37); likewise, for $\omega < 0$, a limiting loop Γ_0' is derived. The poles $\pm\kappa_{\gamma_1}$ also approach the real axis but these do not concern us.

The power to approximate as we did in previous sections depended on our being able to neglect the influence of any other part of the contour when considering the contribution from that part of a loop near its branch point. As the limit of real ω is approached we are no longer justified in doing so, and when the limit is reached and the contour takes the form Γ_0 we have lost entirely the advantage of distinct arcs across which each of λ_{β_1}, etc., is discontinuous. The presence of the factor $\exp\{i\omega t - i\zeta x\}$ in the integrands means, however, that for x large we may still look to the neighbourhood of the branch-points κ_{β_1}, κ_{β_2}, ... to provide the disturbances felt about time $t = x/\beta_1$, $t = x/\beta_2$, ..., respectively. Further, it is now possible to reverse the order of integration and integrate with respect to ω exactly.

This alternative approach will be briefly outlined considering again ${}_p\phi^{(4)}$.

We wish to reinvestigate the contributions from the neighbourhood of the branch points $\pm\kappa_{\beta_1}$, $\pm\kappa_{\beta_2}$ $(\mathscr{R}(\omega) \gtrless 0)$, that is from the portions of Γ_0 denoted in figure 37 by Γ_{0,β_1} and Γ_{0,β_2}. We introduce a new variable v, where

$$\zeta = \kappa_{\beta_1} v, \tag{A. 1}$$

and are therefore interested in $1 \geqslant v \geqslant \beta_1/\alpha_1$.

On Γ_{0,β_1} we have $1 \geqslant v \geqslant \beta_1/\beta_2$, and across the included portion of the cut λ_{β_1} only is discontinuous.

Substituting
$$\left.\begin{aligned} Y &= Y_0 + \lambda_{\beta_1} Y_1, \\ S &= S_0 + \lambda_{\beta_1} S_1, \\ F &= F_0 + \lambda_{\beta_1} F_1, \end{aligned}\right\} \tag{A. 2}$$

where Y_0, S_0, F_0, Y_1, ... do not contain the radical λ_{β_1}, we have

$$\begin{aligned} \frac{1}{F}\left(\frac{Y}{S}\right) &= \frac{Y_0(F_0 S_0 + \lambda_{\beta_1}^2 F_1 S_1) - \lambda_{\beta_1} Y_0(F_1 S_0 + F_0 S_1)}{(F_0^2 - \lambda_{\beta_1}^2 F_1^2)(S_0^2 - \lambda_{\beta_1}^2 S_1^2)}, \\ &= A - \lambda_{\beta_1} B, \quad \text{say,} \end{aligned}$$

since Y_1 is identically zero. The Y_0, S_0, ... here are not to be confused with the Y_{0,β_1}, ..., etc. of the earlier treatment. Here they are functions of ζ, equal to Y_{0,β_1}, ..., etc., at $\zeta = \kappa_{\beta_1}$ but otherwise distinct.

On Γ_{0,β_2} we have $\beta_1/\beta_2 \geqslant v \geqslant \beta_1/\alpha_1$, and across the included portion of the cut λ_{β_1} and λ_{β_2} are discontinuous.

Now writing
$$\left.\begin{aligned} Y &= Y_0 + \lambda_{\beta_1} Y_1 + \lambda_{\beta_2} Y_2, \\ S &= S_0 + \lambda_{\beta_1} S_1 + \lambda_{\beta_2} S_2, \\ F &= F_0 + \lambda_{\beta_1} F_1 + \lambda_{\beta_2} F_2, \end{aligned}\right\} \tag{A. 3}$$

where S_0, S_1, S_2, ... contain λ_{β_1} and λ_{β_2} only in the single-valued combination $(\lambda_{\beta_1}\lambda_{\beta_2})$, we find

$$\begin{aligned} (Y/S)/F = (1/\delta)\,[&Y_0 F_0 S_0 + \lambda_{\beta_1}^2 F_1 S_1 Y_0 + \lambda_{\beta_1}\lambda_{\beta_2}(F_1 S_2 Y_0 - F_1 Y_2 S_0 - Y_2 S_1 F_0) - \lambda_{\beta_2}^2 Y_2 S_2 F_0 \\ &- \lambda_{\beta_1}(F_1 S_0 Y_0 + S_1 F_0 Y_0 + \lambda_{\beta_2}^2 S_2 Y_2 F_0) \\ &- \lambda_{\beta_2}(-Y_2 F_0 S_0 + S_2 F_0 Y_0 - \lambda_{\beta_1}^2 F_1 S_1 Y_0)], \end{aligned} \tag{A. 4}$$

where
$$\delta = (F_0^2 - \lambda_{\beta_1}^2 F_1^2)(S_0^2 - \lambda_{\beta_1}^2 S_1 - 2\lambda_{\beta_1}\lambda_{\beta_2} S_1 S_2 - \lambda_{\beta_2}^2 S_2^2), \tag{A. 5}$$

which we shall write as
$$(Y/S)/F = C - (\lambda_{\beta_1} D + \lambda_{\beta_2} E). \tag{A. 6}$$

From ${}_p\phi^{(4)}$ are then derived displacements

$$ {}_pU^{(4)}_{0,\,\beta_1\beta_2} = \frac{1}{\pi\beta_1}\int_{\beta_1/\alpha_1}^{1} -\frac{8v^2(2v^2-1)\,(1-v^2)^{\frac{1}{2}}}{(v^2-\beta_1^2/\alpha_1^2)^{\frac{1}{2}}} \times\left[M(v)\,\frac{T(P^2-Q^2+T^2)}{(P^2-Q^2+T^2)^2+4P^2Q^2} - N(v)\,\frac{PTQ}{(P^2-Q^2+T^2)^2+4P^2Q^2}\right], \quad \text{(A. 7)} $$

$$ {}_pW^{(4)}_{0,\,\beta_1\beta_2} = \frac{1}{\pi\beta_1}\int_{\beta_1/\alpha_1}^{1} 8v(2v^2-1)\,(1-v^2)^{\frac{1}{2}} \times\left[M(v)\,\frac{P(P^2+T^2+Q^2)}{(P^2-Q^2+T^2)^2+4P^2Q^2} - N(v)\,\frac{Q(P^2+Q^2-T^2)}{(P^2-Q^2+T^2)^2+4P^2Q^2}\right], \quad \text{(A. 8)} $$

where

$$ \left.\begin{aligned} M(v) &= A, \quad \beta_1/\beta_2 \leqslant v \leqslant 1,\\ &= C, \quad \beta_1/\alpha_1 < v < \beta_1/\beta_2, \end{aligned}\right\} \quad \text{(A. 9)} $$

$$ \left.\begin{aligned} N(v) &= (1-v^2)^{\frac{1}{2}}\,B, & \beta_1/\beta_2 \leqslant v \leqslant 1,\\ &= (1-v^2)^{\frac{1}{2}}\,D + (\beta_1^2/\beta_2^2 - v^2)^{\frac{1}{2}}\,E, & \beta_1/\alpha_1 < v < \beta_1/\beta_2, \end{aligned}\right\} \quad \text{(A. 10)} $$

$$ \left.\begin{aligned} P &= (H-h+z)\,(v^2-\beta_1^2/\alpha_1^2)^{\frac{1}{2}}/\beta_1,\\ Q &= H(1-v^2)^{\frac{1}{2}}/\beta_1,\\ T &= t - vx/\beta_1. \end{aligned}\right\} \quad \text{(A. 11)} $$

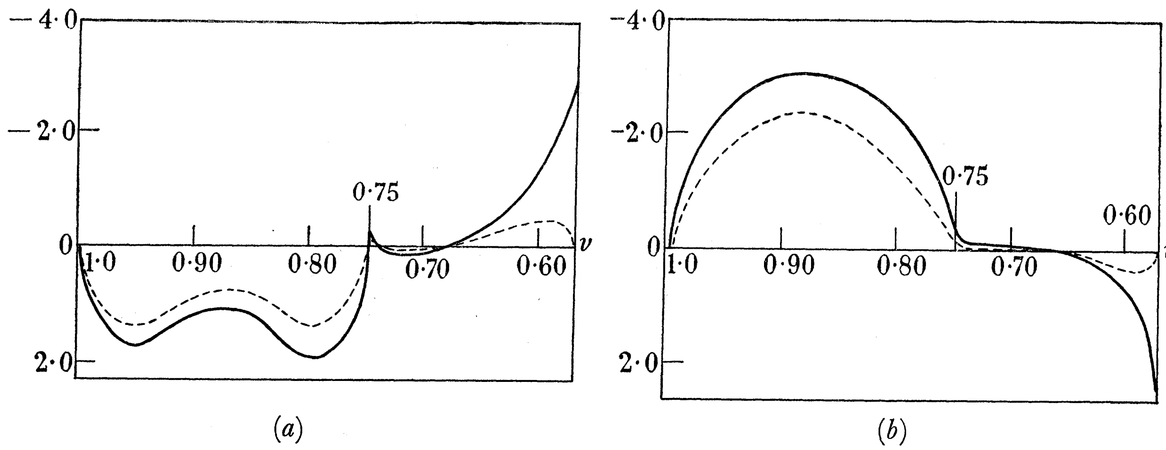

FIGURE 38. Graphs of (a) $M(v)\;8v^2(2v^2-1)\;(1-v^2)^{\frac{1}{2}}/(v^2-\frac{1}{3})^{\frac{1}{2}}$ (full line),
$M(v)\;8v(2v^2-1)\;(1-v^2)^{\frac{1}{2}}$ (dashed line);
(b) $N(v)\;8v^2(2v^2-1)\;(1-v^2)^{\frac{1}{2}}/(v^2-\frac{1}{3})^{\frac{1}{2}}$ (full line),
$N(v)\;8v(2v^2-1)\;(1-v^2)^{\frac{1}{2}}$ (dashed line).

The derivation of these results is sufficiently indicated by Lapwood's treatment of the surface S-pulse' and 'secondary P-pulse', our ${}_p\phi^{(1)}_{\beta_1}$ and ${}_{sp}\phi^{(1)}_{\beta_1}$, respectively (Lapwood 1949, pp. 90 to 94); the essential differences are that there is no second medium and hence no branch-point κ_{β_2} to be considered, the original integrands are simpler and hence the manipulation less cumbersome, and finally one or other of the factors P, Q is always identically zero.

Figure 38 shows the variation with v of the parts of the integrands not containing P, T, Q. The infinite slopes at $v = 1$, $v = \beta_1/\beta_2$ are characteristic of their form; the very small values taken for $v < \beta_1/\beta_2$ arise largely from the choice of elastic constants.

The essential features of the P, T, Q functions are shown in figure 39 (the 'size' of a loop may be roughly estimated from its height and span on the v-axis). It is assumed that x/H is large, so that, by definition of T, a small variation in v will cause a large change in T; in consequence, the functions are small before P, Q have changed appreciably from their values at v_0 defined by $T=0$. Their main behaviour is therefore determined treating P, Q as constant, and we see at once that, to the first order, the third and fourth functions are symmetrical and the first and second asymmetrical about v_0. Only if v_0 coincides with $v=1$ or β_1/α_1, that is P or Q is zero, is the variation of P or Q important.

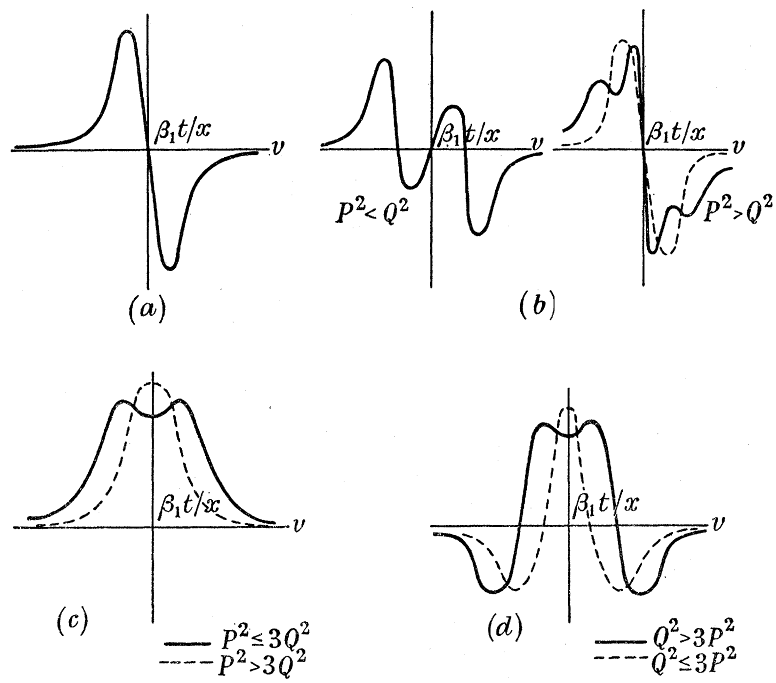

FIGURE 39. Graphs of

$$(a)\quad \frac{PTQ}{(P^2-Q^2+T^2)^2+4P^2Q^2},\qquad (b)\quad \frac{T(P^2-Q^2+T^2)}{(P^2-Q^2+T^2)^2+4P^2Q^2},$$

$$(c)\quad \frac{P(P^2+Q^2+T^2)}{(P^2-Q^2+T^2)^2+4P^2Q^2},\qquad (d)\quad \frac{Q(P^2+Q^2-T^2)}{(P^2-Q^2+T^2)^2+4P^2Q^2}$$

against V.

Now we suppose that each curve in figure 38 is 'multiplied' by the corresponding curve in figure 39 and integrated with respect to v from $v=1$ to some value beyond β_1/β_2. It follows at once, for x large, that at a given time the main contribution comes from the neighbourhood of $v=t\beta_1/x$ ($T=0$) and that for the asymmetrical P, T, Q functions it is inappreciable unless the corresponding $M(v)$, $N(v)$ functions are changing rapidly; for the symmetrical P, T, Q functions, it varies much as the $M(v)$, $N(v)$ functions except near $t=x/\beta_1$, $t=x/\beta_2$.

Each of U, W is given by the sum of two such parts and a qualitative description of each for $t>x/\alpha_1$ is obtained as indicated. Unfortunately, on attempting a graphical addition it becomes clear that the resultant variation of U and W is very sensitive to the relative magnitudes of the two parts, particularly near $t=x/\beta_1$ and $t=x/\beta_2$; without a more quantitative picture of each, it is not found possible to deduce with any confidence the detailed shape of the U, W displacements.

The main conclusion reached is that the variation of $\dot{U}$, $\dot{W}$ is indeed sharper than the approximations of § 9, (*c*), (*d*) would imply, that the decrease from the maximum is almost equally steep in either direction (cf. figures 19 and 20 for the surface S-pulse, ${}_p\phi^{(1)}_{\beta_1}$). The mathematical-graphical process by which this conclusion is reached suggests that it is quite generally true of any of the diffraction pulses.

Otherwise, this alternative approach is found to add little to the methods of part I. When $P = 0$ the integrals do not converge at $v = 1$ ($t = x/\beta_1$), in agreement with the earlier finding (p. 245) that a component for which $h_1 = 0$ does not give the normal diffraction pulse but a sharp minimum-time path pulse like the initial shock.

Appendix 2. Expressions for potentials and displacements corresponding to zero- and first-order terms

Note. (1) When a displacement is written

$$U = f(x,z)\begin{Bmatrix}\mathrm{i}\\1\end{Bmatrix}\left[\frac{T_0}{S_0}\right]_{\beta_2,\,\substack{1,\\0,}} \sin\left(\tfrac{1}{2}\psi \pm \tfrac{1}{4}\pi\right)\cos^{\frac{1}{2}}\psi, \quad \text{say,}$$

it is implied that it is made up of two components, that is,

$$U = f(x,z)\,\{\mathrm{i}[T_0/S_0]_{\beta_2,1}\sin\left(\tfrac{1}{2}\psi \pm \tfrac{1}{4}\pi\right) + [T_0/S_0]_{\beta_2,0}\sin\left(\tfrac{1}{2}\psi - \tfrac{1}{4}\pi\right)\}\cos^{\frac{1}{2}}\psi.$$

(2) When two expressions are bracketed like

$$\begin{cases} U = f_1(x,z,t),\\ \dot{U} = f_2(x,z,t),\end{cases}$$

it is implied that the total displacement is made up of a contribution f_1 and a second contribution whose time-derivative is f_2.

(3) Expressions for potentials as quoted apply when $\mathscr{R}(\omega) > 0$.

Initial and image pulses

$$\begin{cases} \phi_{0,\alpha_1} = \sqrt{\left(\dfrac{2\pi\alpha_1 \mathrm{i}^3}{x\omega}\right)}\exp\{\mathrm{i}\omega\tau\},\\[2ex] U_{0,\alpha_1} = \sqrt{\left(\dfrac{2}{x\alpha_1\tau}\right)}H(\tau), \qquad \tau = t - x/\alpha_1 - (h-z)^2/2x\alpha_1;\\[2ex] W_{0,\alpha_1} = -\dfrac{h-z}{x}\sqrt{\left(\dfrac{2}{x\alpha_1\tau}\right)}H(\tau),\end{cases}$$

$$\begin{cases} \phi_{r,\alpha_1} = -\sqrt{\left(\dfrac{2\pi\alpha_1 \mathrm{i}^3}{x\omega}\right)}\exp\{\mathrm{i}\omega\tau\},\\[2ex] U_{r,\alpha_1} = -\sqrt{\left(\dfrac{2}{x\alpha_1\tau}\right)}H(\tau), \qquad \tau = t - x/\alpha_1 - (h+z)^2/2x\alpha_1;\\[2ex] W_{r,\alpha_1} = -\dfrac{h+z}{x}\sqrt{\left(\dfrac{2}{x\alpha_1\tau}\right)}H(\tau),\end{cases}$$

$$\begin{cases} \psi_{0,\beta_1} = \sqrt{\left(\dfrac{2\pi\beta_1 \mathrm{i}^3}{x\omega}\right)} \exp\{\mathrm{i}\omega\tau\}, \\ U_{0,\beta_1} = -\dfrac{h-z}{x}\sqrt{\left(\dfrac{2}{x\beta_1\tau}\right)} H(\tau), \qquad \tau = t - x/\beta_1 - (h-z)^2/2x\beta_1; \\ W_{0,\beta_1} = -\sqrt{\left(\dfrac{2}{x\beta_1\tau}\right)} H(\tau), \end{cases}$$

$$\begin{cases} \psi_{r,\beta_1} = -\sqrt{\left(\dfrac{2\pi\beta_1 \mathrm{i}^3}{x\omega}\right)} \exp\{\mathrm{i}\omega\tau\}, \\ U_{r,\beta_1} = -\dfrac{h+z}{x}\sqrt{\left(\dfrac{2}{x\beta_1\tau}\right)} H(\tau), \qquad \tau = t - x/\beta_1 - (h+z)^2/2x\beta_1; \\ W_{r,\beta_1} = \sqrt{\left(\dfrac{2}{x\beta_1\tau}\right)} H(\tau). \end{cases}$$

${}_p\phi$ contributions from Γ_{α_2}

$$\begin{cases} {}_p\phi^{(2)}_{\alpha_2} \doteqdot \dfrac{-16\sqrt{(2\pi \mathrm{i}^3)}\,\alpha_2^8(2/\alpha_2^2-1/\beta_1^2)^2}{\widehat{\beta_1\alpha_2}[x\alpha_2 - 2H\widehat{\beta_1\alpha_2} - (h+z)\,\widehat{\alpha_1\alpha_2}]^{\frac{3}{2}}}\left[\dfrac{1}{F_0^2}\left(\dfrac{T_1}{S_0}-\dfrac{T_0S_1}{S_0^2}\right)\right]_{\alpha_2}\omega^{-\frac{3}{2}}\exp\{\mathrm{i}\omega\tau\}, \\ {}_pU^{(2)}_{\alpha_2} \doteqdot \dfrac{-32\sqrt{2}\,\mathrm{i}\,\alpha_2^7(2/\alpha_2^2-1/\beta_1^2)^2}{\widehat{\beta_1\alpha_2}[x\alpha_2 - 2H\widehat{\beta_1\alpha_2} - (h+z)\,\widehat{\alpha_1\alpha_2}]^{\frac{3}{2}}}\left[\dfrac{1}{F_0^2}\left(\dfrac{T_1}{S_0}-\dfrac{T_0S_1}{S_0^2}\right)\right]_{\alpha_2}\tau^{\frac{1}{2}}H(\tau) \doteqdot {}_pW^{(2)}_{\alpha_2}\times\widehat{\alpha_1\alpha_2}/\alpha_2, \end{cases}$$

where $$\tau = t - x/\alpha_2 - 2H/\widehat{\beta_1\alpha_2} - (h+z)/\widehat{\alpha_1\alpha_2};$$

$$\begin{cases} {}_p\phi^{(3)}_{\alpha_2} \doteqdot \dfrac{4\sqrt{(2\pi \mathrm{i})}\,\alpha_2^5\,\widehat{\alpha_1\alpha_2}(2/\alpha_2^2-1/\beta_1^2)}{\widehat{\beta_1\alpha_2}[x\alpha_2 - (H+h-z)\,\widehat{\alpha_1\alpha_2} - H\widehat{\beta_1\alpha_2}]^{\frac{3}{2}}}\left[\dfrac{1}{F_0}\left(\dfrac{Y_1}{S_0}-\dfrac{Y_0S_1}{S_0^2}\right)\right]_{\alpha_2}\omega^{-\frac{3}{2}}\exp\{\mathrm{i}\omega\tau\}, \\ {}_pU^{(3)}_{\alpha_2} \doteqdot \dfrac{8\sqrt{2}\,\alpha_2^4\,\widehat{\alpha_1\alpha_2}(2/\alpha_2^2-1/\beta_1^2)}{\widehat{\beta_1\alpha_2}[x\alpha_2 - (H+h-z)\,\widehat{\alpha_1\alpha_2} - H\widehat{\beta_1\alpha_2}]^{\frac{3}{2}}}\left[\dfrac{1}{F_0}\left(\dfrac{Y_1}{S_0}-\dfrac{Y_0S_1}{S_0^2}\right)\right]_{\alpha_2}\tau^{\frac{1}{2}}H(\tau) \doteqdot -{}_pW^{(3)}_{\alpha_2}\times\widehat{\alpha_1\alpha_2}/\alpha_2, \end{cases}$$

where $$\tau = t - x/\alpha_2 - (H+h-z)/\widehat{\alpha_1\alpha_2} - H/\widehat{\beta_1\alpha_2};$$

$$\begin{cases} {}_p\phi^{(4)}_{\alpha_2} \doteqdot \dfrac{4\sqrt{(2\pi \mathrm{i})}\,\alpha_2^5\,\widehat{\alpha_1\alpha_2}(2/\alpha_2^2-1/\beta_1^2)}{\widehat{\beta_1\alpha_2}[x\alpha_2 - (H-h+z)\,\widehat{\alpha_1\alpha_2} - H\widehat{\beta_1\alpha_2}]^{\frac{3}{2}}}\left[\dfrac{1}{F_0}\left(\dfrac{Y_1}{S_0}-\dfrac{Y_0S_1}{S_0^2}\right)\right]_{\alpha_2}\omega^{-\frac{3}{2}}\exp\{\mathrm{i}\omega\tau\}, \\ {}_pU^{(4)}_{\alpha_2} \doteqdot \dfrac{8\sqrt{2}\,\alpha_2^4\,\widehat{\alpha_1\alpha_2}(2/\alpha_2^2-1/\beta_1^2)}{\widehat{\beta_1\alpha_2}[x\alpha_2 - (H-h+z)\,\widehat{\alpha_1\alpha_2} - H\widehat{\beta_1\alpha_2}]^{\frac{3}{2}}}\left[\dfrac{1}{F_0}\left(\dfrac{Y_1}{S_0}-\dfrac{Y_0S_1}{S_0^2}\right)\right]_{\alpha_2}\tau^{\frac{1}{2}}H(\tau) \doteqdot {}_pW^{(4)}_{\alpha_2}\times\widehat{\alpha_1\alpha_2}/\alpha_2, \end{cases}$$

where $$\tau = t - x/\alpha_2 - (H-h+z)/\widehat{\alpha_1\alpha_2} - H/\widehat{\beta_1\alpha_2};$$

$$\begin{cases} {}_p\phi^{(5)}_{\alpha_2} \doteqdot \dfrac{-8\sqrt{(2\pi \mathrm{i})}\,\alpha_2^5\,\widehat{\alpha_1\alpha_2}(2/\alpha_2^2-1/\beta_1^2)}{\widehat{\beta_1\alpha_2}[x\alpha_2 - (H+h+z)/\widehat{\alpha_1\alpha_2} - H\widehat{\beta_1\alpha_2}]^{\frac{3}{2}}}\left[\dfrac{\overline{F}_0}{F_0^2}\left(\dfrac{Y_1}{S_0}-\dfrac{Y_0S_1}{S_0^2}\right)\right]_{\alpha_2}\omega^{-\frac{3}{2}}\exp\{\mathrm{i}\omega\tau\}, \\ {}_pU^{(5)}_{\alpha_2} \doteqdot \dfrac{-16\sqrt{2}\,\alpha_2^4\,\widehat{\alpha_1\alpha_2}(2/\alpha_2^2-1/\beta_1^2)}{\widehat{\beta_1\alpha_2}[x\alpha_2 - (H+h+z)/\widehat{\alpha_1\alpha_2} - H\widehat{\beta_1\alpha_2}]^{\frac{3}{2}}}\left[\dfrac{\overline{F}_0}{F_0^2}\left(\dfrac{Y_1}{S_0}-\dfrac{Y_0S_1}{S_0^2}\right)\right]_{\alpha_2}\tau^{\frac{1}{2}}H(\tau) \doteqdot {}_pW^{(5)}_{\alpha_2}\times\widehat{\alpha_1\alpha_2}/\alpha_2, \end{cases}$$

where $$\tau = t - x/\alpha_2 - (H+h+z)/\widehat{\alpha_1\alpha_2} - H/\widehat{\beta_1\alpha_2};$$

$$\begin{cases} {}_p\phi^{(6)}_{\alpha_2} \doteqdot \surd(2\pi i^3)\, \widehat{\alpha_1\alpha_2}\, \alpha_2^2[x\alpha_2-(2H-h-z)\,\widehat{\alpha_1\alpha_2}]^{-\frac{3}{2}} \left[\frac{W_1}{S_0}-\frac{W_0S_1}{S_0^2}\right]_{\alpha_2} \omega^{-\frac{3}{2}} \exp\{i\omega\tau\}, \\ {}_pU^{(6)}_{\alpha_2} \doteqdot 2\surd 2\, i\, \widehat{\alpha_1\alpha_2}\, \alpha_2[x\alpha_2-(2H-h-z)\,\widehat{\alpha_1\alpha_2}]^{-\frac{3}{2}} \left[\frac{W_1}{S_0}-\frac{W_0S_1}{S_0^2}\right]_{\alpha_2} \tau^{\frac{1}{2}}H(\tau) \doteqdot -{}_pW^{(6)}_{\alpha_2}\times\widehat{\alpha_1\alpha_2}/\alpha_2, \end{cases}$$

where $$\tau = t-x/\alpha_2-(2H-h-z)/\widehat{\alpha_1\alpha_2};$$

$$\begin{cases} {}_p\phi^{(7)}_{\alpha_2} \doteqdot -\surd(2\pi i^3)\, \widehat{\alpha_1\alpha_2}\, \alpha_2^2[x\alpha_2-(2H+h-z)\,\widehat{\alpha_1\alpha_2}]^{-\frac{3}{2}} \left[\frac{\overline{F}_0}{F_0}\left(\frac{W_1}{S_0}-\frac{W_0S_1}{S_0^2}\right)\right]_{\alpha_2} \omega^{-\frac{3}{2}} \exp\{i\omega\tau\}, \\ {}_pU^{(7)}_{\alpha_2} \doteqdot -2\surd 2\, i\, \widehat{\alpha_1\alpha_2}\, \alpha_2[x\alpha_2-(2H+h-z)\,\widehat{\alpha_1\alpha_2}]^{-\frac{3}{2}} \left[\frac{\overline{F}_0}{F_0}\left(\frac{W_1}{S_0}-\frac{W_0S_1}{S_0^2}\right)\right]_{\alpha_2} \tau^{\frac{1}{2}}H(\tau) \doteqdot -{}_pW^{(7)}_{\alpha_2}\times\widehat{\alpha_1\alpha_2}/\alpha_2 \end{cases}$$

where $$\tau = t-x/\alpha_2-(2H+h-z)/\widehat{\alpha_1\alpha_2};$$

$$\begin{cases} {}_p\phi^{(8)}_{\alpha_2} \doteqdot -\surd(2\pi i^3)\, \widehat{\alpha_1\alpha_2}\, \alpha_2^2[x\alpha_2-(2H-h+z)\,\widehat{\alpha_1\alpha_2}]^{-\frac{3}{2}} \left[\frac{\overline{F}_0}{F_0}\left(\frac{W_1}{S_0}-\frac{W_0S_1}{S_0^2}\right)\right]_{\alpha_2} \omega^{-\frac{3}{2}} \exp\{i\omega\tau\}, \\ {}_pU^{(8)}_{\alpha_2} \doteqdot -2\surd 2\, i\, \widehat{\alpha_1\alpha_2}\, \alpha_2[x\alpha_2-(2H-h+z)\,\widehat{\alpha_1\alpha_2}]^{-\frac{3}{2}} \left[\frac{\overline{F}_0}{F_0}\left(\frac{W_1}{S_0}-\frac{W_0S_1}{S_0^2}\right)\right]_{\alpha_2} \tau^{\frac{1}{2}}H(\tau) \doteqdot {}_pW^{(8)}_{\alpha_2}\times\widehat{\alpha_1\alpha_2}/\alpha_2, \end{cases}$$

where $$\tau = t-x/\alpha_2-(2H-h+z)/\widehat{\alpha_1\alpha_2};$$

$$\begin{cases} {}_p\phi^{(9)}_{\alpha_2} = \surd(2\pi i^3)\, \widehat{\alpha_1\alpha_2}\, \alpha_2^2[x\alpha_2-(2H+h+z)\,\widehat{\alpha_1\alpha_2}]^{-\frac{3}{2}} \left[\frac{\overline{F}_0^2}{F_0^2}\left(\frac{W_1}{S_0}-\frac{W_0S_1}{S_0^2}\right)\right]_{\alpha_2} \omega^{-\frac{3}{2}} \exp\{i\omega\tau\}, \\ {}_pU^{(9)}_{\alpha_2} = 2\surd 2\, i\, \widehat{\alpha_1\alpha_2}\, \alpha_2[x\alpha_2-(2H+h+z)\,\widehat{\alpha_1\alpha_2}]^{-\frac{3}{2}} \left[\frac{\overline{F}_0^2}{F_0^2}\left(\frac{W_1}{S_0}-\frac{W_0S_1}{S_0^2}\right)\right]_{\alpha_2} \tau^{\frac{1}{2}}H(\tau) \doteqdot {}_pW^{(9)}_{\alpha_2}\times\widehat{\alpha_1\alpha_2}/\alpha_2, \end{cases}$$

where $$\tau = t-x/\alpha_2-(2H+h+z)/\widehat{\alpha_1\alpha_2}.$$

${}_p\psi$ contributions from Γ_{α_2}

$$\begin{cases} {}_p\psi^{(2)}_{\alpha_2} \doteqdot \dfrac{-4\surd(2\pi i^3)\,\alpha_2^5(2/\alpha_2^2-1/\beta_1^2)}{[x\alpha_2-(2H-z)\,\widehat{\beta_1\alpha_2}-h\widehat{\alpha_1\alpha_2}]^{\frac{3}{2}}} \left[\frac{1}{F_0}\left(\frac{T_1}{S_0}-\frac{T_0S_1}{S_0^2}\right)\right]_{\alpha_2} \omega^{-\frac{3}{2}}\exp\{i\omega\tau\}, \\ {}_pU^{(2)}_{\alpha_2} \doteqdot \dfrac{8\surd 2\, i\,\alpha_2^5(2/\alpha_2^2-1/\beta_1^2)}{\widehat{\beta_1\alpha_2}[x\alpha_2-(2H-z)\,\widehat{\beta_1\alpha_2}-h\widehat{\alpha_1\alpha_2}]^{\frac{3}{2}}} \left[\frac{1}{F_0}\left(\frac{T_1}{S_0}-\frac{T_0S_1}{S_0^2}\right)\right]_{\alpha_2} \tau^{\frac{1}{2}}H(\tau) \doteqdot -{}_pW^{(2)}_{\alpha_2}\times\alpha_2/\widehat{\beta_1\alpha_2}, \end{cases}$$

where $$\tau = t-x/\alpha_2-(2H-z)/\widehat{\beta_1\alpha_2}-h/\widehat{\alpha_1\alpha_2};$$

$$\begin{cases} {}_p\psi^{(3)}_{\alpha_2} \doteqdot \dfrac{4\surd(2\pi i^3)\,\alpha_2^5(2/\alpha_2^2-1/\beta_1^2)}{[x\alpha_2-(2H+z)\,\widehat{\beta_1\alpha_2}-h\widehat{\alpha_1\alpha_2}]^{\frac{3}{2}}} \left[\frac{\overline{F}_0}{F_0^2}\left(\frac{T_1}{S_0}-\frac{T_0S_1}{S_0^2}\right)\right]_{\alpha_2} \omega^{-\frac{3}{2}}\exp\{i\omega\tau\}, \\ {}_pU^{(3)}_{\alpha_2} \doteqdot \dfrac{8\surd 2\, i\,\alpha_2^5(2/\alpha_2^2-1/\beta_1^2)}{\widehat{\beta_1\alpha_2}[x\alpha_2-(2H+z)\,\widehat{\beta_1\alpha_2}-h\widehat{\alpha_1\alpha_2}]^{\frac{3}{2}}} \left[\frac{\overline{F}_0}{F_0^2}\left(\frac{T_1}{S_0}-\frac{T_0S_1}{S_0^2}\right)\right]_{\alpha_2} \tau^{\frac{1}{2}}H(\tau) \doteqdot {}_pW^{(3)}_{\alpha_2}\times\alpha_2/\widehat{\beta_1\alpha_2}, \end{cases}$$

where $$\tau = t-x/\alpha_2-(2H+z)/\widehat{\beta_1\alpha_2}-h/\widehat{\alpha_1\alpha_2};$$

$$\begin{cases} {}_p\psi^{(4)}_{\alpha_2} \doteqdot \dfrac{\surd(2\pi i)\,\alpha_2^2\,\widehat{\alpha_1\alpha_2}}{[x\alpha_2-(H-h)\,\widehat{\alpha_1\alpha_2}-(H-z)\,\widehat{\beta_1\alpha_2}]^{\frac{3}{2}}} \left[\frac{Y_1}{S_0}-\frac{Y_0S_1}{S_0^2}\right]_{\alpha_2} \omega^{-\frac{3}{2}}\exp\{i\omega\tau\}, \\ {}_pU^{(4)}_{\alpha_2} \doteqdot \dfrac{-2\surd 2\,\widehat{\alpha_1\alpha_2}\,\alpha_2^2}{\widehat{\beta_1\alpha_2}[x\alpha_2-(H-h)\,\widehat{\alpha_1\alpha_2}-(H-z)\,\widehat{\beta_1\alpha_2}]^{\frac{3}{2}}} \left[\frac{Y_1}{S_0}-\frac{Y_0S_1}{S_0^2}\right]_{\alpha_2} \tau^{\frac{1}{2}}H(\tau) \doteqdot -{}_pW^{(4)}_{\alpha_2}\times\alpha_2/\widehat{\beta_1\alpha_2}, \end{cases}$$

where $$\tau = t-x/\alpha_2-(H-h)/\widehat{\alpha_1\alpha_2}-(H-z)\,\widehat{\beta_1\alpha_2};$$

$$\begin{cases} {}_p\psi^{(5)}_{\alpha_2} \doteqdot \dfrac{-\sqrt{(2\pi \mathrm{i})}\,\alpha_2^2\,\widehat{\alpha_1\alpha_2}}{[x\alpha_2-(H-h)\,\widehat{\alpha_1\alpha_2}-(H+z)\,\widehat{\beta_1\alpha_2}]^{\frac{3}{2}}}\left[\dfrac{\overline{F}_0}{F_0}\left(\dfrac{Y_1}{S_0}-\dfrac{Y_0S_1}{S_0^2}\right)\right]_{\alpha_2}\omega^{-\frac{3}{2}}\exp\{\mathrm{i}\omega\tau\}, \\ {}_pU^{(5)}_{\alpha_2} \doteqdot \dfrac{-2\sqrt{2}\,\widehat{\alpha_1\alpha_2}\,\alpha_2^2}{\widehat{\beta_1\alpha_2}[x\alpha_2-(H-h)\,\widehat{\alpha_1\alpha_2}-(H+z)\,\widehat{\beta_1\alpha_2}]^{\frac{3}{2}}}\left[\dfrac{\overline{F}_0}{F_0}\left(\dfrac{Y_1}{S_0}-\dfrac{Y_0S_1}{S_0^2}\right)\right]_{\alpha_2}\tau^{\frac{1}{2}}H(\tau) \doteqdot {}_pW^{(5)}_{\alpha_2}\times\alpha_2/\widehat{\beta_1\alpha_2}, \end{cases}$$

where $$\tau = t-x/\alpha_2-(H-h)/\widehat{\alpha_1\alpha_2}-(H+z)/\widehat{\beta_1\alpha_2};$$

$$\begin{cases} {}_p\psi^{(6)}_{\alpha_2} \doteqdot \dfrac{-\sqrt{(2\pi \mathrm{i})}\,\alpha_2^2\,\widehat{\alpha_1\alpha_2}}{[x\alpha_2-(H+h)\,\widehat{\alpha_1\alpha_2}-(H-z)\,\widehat{\beta_1\alpha_2}]^{\frac{3}{2}}}\left[\dfrac{\overline{F}_0}{F_0}\left(\dfrac{Y_1}{S_0}-\dfrac{Y_0S_1}{S_0^2}\right)\right]_{\alpha_2}\omega^{-\frac{3}{2}}\exp\{\mathrm{i}\omega\tau\}, \\ {}_pU^{(6)}_{\alpha_2} \doteqdot \dfrac{2\sqrt{2}\,\widehat{\alpha_1\alpha_2}\,\alpha_2^2}{\widehat{\beta_1\alpha_2}[x\alpha_2-(H+h)\,\widehat{\alpha_1\alpha_2}-(H-z)\,\widehat{\beta_1\alpha_2}]^{\frac{3}{2}}}\left[\dfrac{\overline{F}_0}{F_0}\left(\dfrac{Y_1}{S_0}-\dfrac{Y_0S_1}{S_0^2}\right)\right]_{\alpha_2}\tau^{\frac{1}{2}}H(\tau) \doteqdot -{}_pW^{(6)}_{\alpha_2}\times\alpha_2/\widehat{\beta_1\alpha_2}, \end{cases}$$

where $$\tau = t-x/\alpha_2-(H+h)/\widehat{\alpha_1\alpha_2}-(H-z)/\widehat{\beta_1\alpha_2};$$

$$\begin{cases} {}_p\psi^{(7)}_{\alpha_2} \doteqdot \dfrac{\sqrt{(2\pi \mathrm{i})}\,\alpha_2^2\,\widehat{\alpha_1\alpha_2}}{[x\alpha_2-(H+h)\,\widehat{\alpha_1\alpha_2}-(H+z)\,\widehat{\beta_1\alpha_2}]^{\frac{3}{2}}}\left[\left(\dfrac{\overline{F}_0^2}{F_0^2}-\dfrac{16\alpha_2^6(2/\alpha_2^2-1/\beta_1^2)^2}{\widehat{\alpha_1\alpha_2}\,\widehat{\beta_1\alpha_2}\,F_0^2}\right)\left(\dfrac{Y_1}{S_0}-\dfrac{Y_0S_1}{S_0^2}\right)\right]_{\alpha_2}\omega^{-\frac{3}{2}}\exp\{\mathrm{i}\omega\tau\}, \\ {}_pU^{(7)}_{\alpha_2} \doteqdot \dfrac{2\sqrt{2}\,\widehat{\alpha_1\alpha_2}\,\alpha_2^2}{\widehat{\beta_1\alpha_2}[x\alpha_2-(H+h)\,\widehat{\alpha_1\alpha_2}-(H+z)\,\widehat{\beta_1\alpha_2}]^{\frac{3}{2}}}\left[\left(\dfrac{\overline{F}_0^2}{F_0^2}-\dfrac{16\alpha_2^6(2/\alpha_2^2-1/\beta_1^2)^2}{\widehat{\alpha_1\alpha_2}\,\widehat{\beta_1\alpha_2}\,F_0^2}\right)\left(\dfrac{Y_1}{S_0}-\dfrac{Y_0S_1}{S_0^2}\right)\right]_{\alpha_2}\tau^{\frac{1}{2}}H(\tau) \\ \qquad \doteqdot {}_pW^{(7)}_{\alpha_2}\times\alpha_2/\widehat{\beta_1\alpha_2}, \end{cases}$$

where $$\tau = t-x/\alpha_2-(H+h)/\widehat{\alpha_1\alpha_2}-(H+z)/\widehat{\beta_1\alpha_2};$$

$$\begin{cases} {}_p\psi^{(8)}_{\alpha_2} \doteqdot \dfrac{-4\sqrt{(2\pi \mathrm{i}^3)}\,\alpha_2^5(2/\alpha_2^2-1/\beta_1^2)}{[x\alpha_2-(2H-h)\,\widehat{\alpha_1\alpha_2}-z\widehat{\beta_1\alpha_2}]^{\frac{3}{2}}}\left[\dfrac{1}{F_0}\left(\dfrac{W_1}{S_0}-\dfrac{W_0S_1}{S_0^2}\right)\right]_{\alpha_2}\omega^{-\frac{3}{2}}\exp\{\mathrm{i}\omega\tau\}, \\ {}_pU^{(8)}_{\alpha_2} \doteqdot \dfrac{-8\sqrt{2}\,\mathrm{i}\,\alpha_2^5(2/\alpha_2^2-1/\beta_1^2)}{\widehat{\beta_1\alpha_2}[x\alpha_2-(2H-h)\,\widehat{\alpha_1\alpha_2}-z\widehat{\beta_1\alpha_2}]^{\frac{3}{2}}}\left[\dfrac{1}{F_0}\left(\dfrac{W_1}{S_0}-\dfrac{W_0S_1}{S_0^2}\right)\right]_{\alpha_2}\tau^{\frac{1}{2}}H(\tau) \doteqdot {}_pW^{(8)}_{\alpha_2}\times\alpha_2/\widehat{\beta_1\alpha_2}, \end{cases}$$

where $$\tau = t-x/\alpha_2-(2H-h)/\widehat{\alpha_1\alpha_2}-z/\widehat{\beta_1\alpha_2};$$

$$\begin{cases} {}_p\psi^{(9)}_{\alpha_2} \doteqdot \dfrac{4\sqrt{(2\pi \mathrm{i}^3)}\,\alpha_2^5(2/\alpha_2^2-1/\beta_1^2)}{[x\alpha_2-(2H+h)\,\widehat{\alpha_1\alpha_2}-z\widehat{\beta_1\alpha_2}]^{\frac{3}{2}}}\left[\dfrac{\overline{F}_0}{F_0^2}\left(\dfrac{W_1}{S_0}-\dfrac{W_0S_1}{S_0^2}\right)\right]_{\alpha_2}\omega^{-\frac{3}{2}}\exp\{\mathrm{i}\omega\tau\}, \\ {}_pU^{(9)}_{\alpha_2} \doteqdot \dfrac{8\sqrt{2}\,\mathrm{i}\,\alpha_2^5(2/\alpha_2^2-1/\beta_1^2)}{\widehat{\beta_1\alpha_2}[x\alpha_2-(2H+h)\,\widehat{\alpha_1\alpha_2}-z\widehat{\beta_1\alpha_2}]^{\frac{3}{2}}}\left[\dfrac{\overline{F}_0}{F_0}\left(\dfrac{W_1}{S_0}-\dfrac{W_0S_1}{S_0^2}\right)\right]_{\alpha_2}\tau^{\frac{1}{2}}H(\tau) \doteqdot {}_pW^{(9)}_{\alpha_2}\times\alpha_2/\widehat{\beta_1\alpha_2}, \end{cases}$$

where $$\tau = t-x/\alpha_2-(2H+h)/\widehat{\alpha_1\alpha_2}-z/\widehat{\beta_1\alpha_2}.$$

${}_p\phi$ contributions from Γ_{α_1}

$$\begin{cases} {}_p\phi^{(1)}_{\alpha_1} \doteqdot 8\sqrt{(2\pi \mathrm{i}^3)}\,\alpha_1^{-\frac{5}{2}}\,\widehat{\beta_1\alpha_1}^{-1}(2/\alpha_1^2-1/\beta_1^2)^{-2}x^{-\frac{3}{2}}(h+z)\,\omega^{-\frac{1}{2}}\exp\{\mathrm{i}\omega\tau\}, \\ {}_pU^{(1)}_{\alpha_1} \doteqdot 8\sqrt{2}\,\alpha_1^{-\frac{7}{2}}\,\widehat{\beta_1\alpha_1}^{-1}(2/\alpha_1^2-1/\beta_1^2)^{-2}x^{-\frac{3}{2}}(h+z)\,\tau^{-\frac{1}{2}}H(\tau), \\ {}_pW^{(1)}_{\alpha_1} \doteqdot -16\sqrt{2}\,\alpha_1^{-\frac{5}{2}}\,\widehat{\beta_1\alpha_1}^{-1}(2/\alpha_1^2-1/\beta_1^2)^{-2}x^{-\frac{3}{2}}[\tau^{\frac{1}{2}}-\tfrac{1}{2}(h+z)^2/x\alpha_1\tau^{\frac{1}{2}}]\,H(\tau), \end{cases}$$

where $$\tau = t-x/\alpha_1-\tfrac{1}{2}(h+z)^2/x\alpha_1;$$

$$\left\{\begin{aligned}
{}_p\phi^{(2)}_{\alpha_1} &\doteqdot \frac{16\sqrt{(2\pi\mathrm{i}^3)}\,(x\alpha_1-2H\widehat{\beta_1\alpha_1})^{-\frac{3}{2}}}{\alpha_1\widehat{\beta_1\alpha_1}(2/\alpha_1^2-1/\beta_1^2)^2}\left[\frac{T_0}{S_0}\right]_{\alpha_1}(h+z)\,\omega^{-\frac{1}{2}}\exp\{\mathrm{i}\omega\tau\},\\
{}_pU^{(2)}_{\alpha_1} &\doteqdot \frac{16\sqrt{2}\,(x\alpha_1-2H\widehat{\beta_1\alpha_1})^{-\frac{3}{2}}}{\alpha_1^2\widehat{\beta_1\alpha_1}(2/\alpha_1^2-1/\beta_1^2)^2}\left[\frac{T_0}{S_0}\right]_{\alpha_1,0}(h+z)\,\tau^{-\frac{1}{2}}H(\tau),\\
{}_p\dot{U}^{(2)}_{\alpha_1} &\doteqdot \frac{-8\sqrt{2}\,\mathrm{i}\,(x\alpha_1-2H\widehat{\beta_1\alpha_1})^{-\frac{3}{2}}}{\alpha_1^2\widehat{\beta_1\alpha_1}(2/\alpha_1^2-1/\beta_1^2)^2}\left[\frac{T_0}{S_0}\right]_{\alpha_1,1}(h+z)\,\tau'^{-\frac{3}{2}}H(\tau'),\\
{}_pW^{(2)}_{\alpha_1} &\doteqdot \frac{-32\sqrt{2}\,(x\alpha_1-2H\widehat{\beta_1\alpha_1})^{-\frac{3}{2}}}{\alpha_1\widehat{\beta_1\alpha_1}(2/\alpha_1^2-1/\beta_1^2)^2}\left[\frac{T_0}{S_0}\right]_{\alpha_1,0}[\tau^{\frac{1}{2}}-\tfrac{1}{2}(h+z)^2/(x\alpha_1-2H\widehat{\beta_1\alpha_1})\,\tau^{\frac{1}{2}}]\,H(\tau),\\
{}_p\dot{W}^{(2)}_{\alpha_1} &\doteqdot \frac{16\sqrt{2}\,\mathrm{i}\,(x\alpha_1-2H\widehat{\beta_1\alpha_1})^{-\frac{3}{2}}}{\alpha_1\widehat{\beta_1\alpha_1}(2/\alpha_1^2-1/\beta_1^2)^2}\left[\frac{T_0}{S_0}\right]_{\alpha_1,1}[\tau'^{-\frac{1}{2}}-\tfrac{1}{2}(h+z)^2/(x\alpha_1-2H\widehat{\beta_1\alpha_1})\,\tau'^{\frac{3}{2}}]\,H(\tau'),
\end{aligned}\right.$$

where $\tau=-\tau'=t-x/\alpha_1-2H/\widehat{\beta_1\alpha_1}-(h+z)^2/2(x\alpha_1-2H\widehat{\beta_1\alpha_1})$;

$$\left\{\begin{aligned}
{}_p\phi^{(3)}_{\alpha_1} &\doteqdot \frac{-4\sqrt{(2\pi\mathrm{i}^3)}\,\alpha_1(x\alpha_1-H\widehat{\beta_1\alpha_1})^{-\frac{3}{2}}}{\widehat{\beta_1\alpha_1}(2/\alpha_1^2-1/\beta_1^2)}\left[\frac{Y_1}{S_0}\right]_{\alpha_1}(H+h-z)\,\omega^{-\frac{1}{2}}\exp\{\mathrm{i}\omega\tau\},\\
{}_pU^{(3)}_{\alpha_1} &\doteqdot \frac{-4\sqrt{2}\,(x\alpha_1-H\widehat{\beta_1\alpha_1})^{-\frac{3}{2}}}{\widehat{\beta_1\alpha_1}(2/\alpha_1^2-1/\beta_1^2)}\left[\frac{Y_1}{S_0}\right]_{\alpha_1,0}(H+h-z)\,\tau^{-\frac{1}{2}}H(\tau),\\
{}_p\dot{U}^{(3)}_{\alpha_1} &\doteqdot \frac{2\sqrt{2}\,\mathrm{i}\,(x\alpha_1-H\widehat{\beta_1\alpha_1})^{-\frac{3}{2}}}{\widehat{\beta_1\alpha_1}(2/\alpha_1^2-1/\beta_1^2)}\left[\frac{Y_1}{S_0}\right]_{\alpha_1,1}(H+h-z)\,\tau'^{-\frac{3}{2}}H(\tau'),\\
{}_pW^{(3)}_{\alpha_1} &\doteqdot \frac{-8\sqrt{2}\,\alpha_1(x\alpha_1-H\widehat{\beta_1\alpha_1})^{-\frac{3}{2}}}{\widehat{\beta_1\alpha_1}(2/\alpha_1^2-1/\beta_1^2)}\left[\frac{Y_1}{S_0}\right]_{\alpha_1,0}[\tau^{\frac{1}{2}}-\tfrac{1}{2}(H+h-z)^2/(x\alpha_1-H\widehat{\beta_1\alpha_1})\,\tau^{\frac{1}{2}}]\,H(\tau),\\
{}_p\dot{W}^{(3)}_{\alpha_1} &\doteqdot \frac{4\sqrt{2}\,\mathrm{i}\,\alpha_1(x\alpha_1-H\widehat{\beta_1\alpha_1})^{-\frac{3}{2}}}{\widehat{\beta_1\alpha_1}(2/\alpha_1^2-1/\beta_1^2)}\left[\frac{Y_1}{S_0}\right]_{\alpha_1,1}[\tau'^{-\frac{1}{2}}-\tfrac{1}{2}(H+h-z)^2/(x\alpha_1-H\widehat{\beta_1\alpha_1})\,\tau'^{-\frac{3}{2}}]\,H(\tau'),
\end{aligned}\right.$$

where $\tau=-\tau'=t-x/\alpha_1-H/\widehat{\beta_1\alpha_1}-\tfrac{1}{2}(H+h-z)^2/(x\alpha_1-H\widehat{\beta_1\alpha_1})$;

and similarly

$${}_p\phi^{(4)}_{\alpha_1}\doteqdot(+1)\times\ldots,\quad {}_pU^{(4)}_{\alpha_1},\ {}_p\dot{U}^{(4)}_{\alpha_1}\doteqdot(+1)\times\ldots,\quad {}_pW^{(4)}_{\alpha_1},\ {}_p\dot{W}^{(4)}_{\alpha_1}\doteqdot(-1)\times\ldots$$

with $\mp\tfrac{1}{2}(H-h+z)^2$ for $\tfrac{1}{2}(H+h-z)^2$ in $\left\{\begin{array}{l}\text{bracketed expression}\\ \tau,\ -\tau'\end{array}\right.$;

$${}_p\phi^{(5)}_{\alpha_1}\doteqdot(-2)\times\ldots,\quad {}_pU^{(5)}_{\alpha_1},\ {}_p\dot{U}^{(5)}_{\alpha_1}\doteqdot(-2)\times\ldots,\quad {}_pW^{(5)}_{\alpha_1},\ {}_p\dot{W}^{(5)}_{\alpha_1}\doteqdot(+2)\times\ldots$$

with $\mp\tfrac{1}{2}(H+h+z)^2$ for $\tfrac{1}{2}(H+h-z)^2$ in $\left\{\begin{array}{l}\text{bracketed expression}\\ \tau,\ -\tau'\end{array}\right.$;

$$\left\{\begin{aligned}
{}_p\phi^{(6)}_{\alpha_1} &\doteqdot -\sqrt{(2\pi\alpha_1\mathrm{i}^3)}\,x^{-\frac{1}{2}}\omega^{-\frac{1}{2}}\exp\{\mathrm{i}\omega\tau\},\\
{}_pU^{(6)}_{\alpha_1} &\doteqdot -\sqrt{(2/\alpha_1 x)}\,\tau^{-\frac{1}{2}}H(\tau),\\
{}_pW^{(6)}_{\alpha_1} &\doteqdot \sqrt{(2/\alpha_1 x^3)}\,(2H-h-z)\,\tau^{-\frac{1}{2}}H(\tau),
\end{aligned}\right.$$

where $\tau=t-x/\alpha_1-(2H-h-z)^2/2x\alpha_1$;

and similarly $\quad {}_p\phi^{(7)}_{\alpha_1} \doteqdot (-1)\times\ldots, \quad {}_pU^{(7)}_{\alpha_1} \doteqdot (-1)\times\ldots, \quad {}_pW^{(7)}_{\alpha_1} \doteqdot (-1)\times\ldots$

with $\qquad (2H+h-z) \quad \text{for} \quad (2H-h-z),$

$${}_p\phi^{(8)}_{\alpha_1} \doteqdot (-1)\times\ldots, \quad {}_pU^{(8)}_{\alpha_1} \doteqdot (-1)\times\ldots, \quad {}_pW^{(8)}_{\alpha_1} \doteqdot (+1)\times\ldots$$

with $\qquad (2H-h+z) \quad \text{for} \quad (2H-h-z),$

$${}_p\phi^{(9)}_{\alpha_1} \doteqdot (+1)\times\ldots, \quad {}_pU^{(9)}_{\alpha_1} \doteqdot (+1)\times\ldots, \quad {}_pW^{(9)}_{\alpha_1} \doteqdot (-1)\times\ldots$$

with $\qquad (2H+h+z) \quad \text{for} \quad (2H-h-z).$

${}_p\psi$ contributions from Γ_{α_1}

$$\begin{cases} {}_p\psi^{(1)}_{\alpha_1} \doteqdot \dfrac{-4\sqrt{(2\pi \mathrm{i}^3)}\,(x\alpha_1 - z\widehat{\beta_1\alpha_1})^{-\frac{3}{2}}}{(2/\alpha_1^2 - 1/\beta_1^2)}\, h\omega^{-\frac{1}{2}} \exp\{\mathrm{i}\omega\tau\}, \\[2ex] {}_pU^{(1)}_{\alpha_1} \doteqdot \dfrac{-4\sqrt{2}\,(x\alpha_1 - z\widehat{\beta_1\alpha_1})^{-\frac{3}{2}}}{\widehat{\beta_1\alpha_1}(2/\alpha_1^2 - 1/\beta_1^2)}\, h\tau^{-\frac{1}{2}} H(\tau), \\[2ex] {}_pW^{(1)}_{\alpha_1} \doteqdot \dfrac{4\sqrt{2}\,(x\alpha_1 - z\widehat{\beta_1\alpha_1})^{-\frac{3}{2}}}{\alpha_1(2/\alpha_1^2 - 1/\beta_1^2)}\, h\tau^{-\frac{1}{2}} H(\tau), \end{cases}$$

where $\qquad \tau = t - x/\alpha_1 - z/\widehat{\beta_1\alpha_1} - h^2/2(x\alpha_1 - z\widehat{\beta_1\alpha_1});$

$$\begin{cases} {}_p\psi^{(2)}_{\alpha_1} \doteqdot \dfrac{4\sqrt{(2\pi \mathrm{i}^3)}\,[x\alpha_1 - (2H-z)\,\widehat{\beta_1\alpha_1}]^{-\frac{3}{2}}}{(2/\alpha_1^2 - 1/\beta_1^2)} \left[\dfrac{T_0}{S_0}\right]_{\alpha_1} h\omega^{-\frac{1}{2}} \exp\{\mathrm{i}\omega\tau\}, \\[2ex] {}_pU^{(2)}_{\alpha_1} \doteqdot \dfrac{-4\sqrt{2}\,[x\alpha_1 - (2H-z)\,\widehat{\beta_1\alpha_1}]^{-\frac{3}{2}} h}{\widehat{\beta_1\alpha_1}(2/\alpha_1^2 - 1/\beta_1^2)} \left[\dfrac{T_0}{S_0}\right]_{\alpha_1, 0} \tau^{-\frac{1}{2}} H(\tau), \\[2ex] {}_p\dot{U}^{(2)}_{\alpha_1} \doteqdot \dfrac{2\sqrt{2}\,\mathrm{i}\,[x\alpha_1 - (2H-z)\,\widehat{\beta_1\alpha_1}]^{-\frac{3}{2}} h}{\widehat{\beta_1\alpha_1}(2/\alpha_1^2 - 1/\beta_1^2)} \left[\dfrac{T_0}{S_0}\right]_{\alpha_1, 1} \tau'^{-\frac{3}{2}} H(\tau'), \\[2ex] {}_pW^{(2)}_{\alpha_1} \doteqdot \dfrac{-4\sqrt{2}\,[x\alpha_1 - (2H-z)\,\widehat{\beta_1\alpha_1}]^{-\frac{3}{2}} h}{\alpha_1(2/\alpha_1^2 - 1/\beta_1^2)} \left[\dfrac{T_0}{S_0}\right]_{\alpha_1, 0} \tau^{-\frac{1}{2}} H(\tau), \\[2ex] {}_p\dot{W}^{(2)}_{\alpha_1} \doteqdot \dfrac{2\sqrt{2}\,\mathrm{i}\,[x\alpha_1 - (2H-z)\,\widehat{\beta_1\alpha_1}]^{-\frac{3}{2}} h}{\alpha_1(2/\alpha_1^2 - 1/\beta_1^2)} \left[\dfrac{T_0}{S_0}\right]_{\alpha_1, 1} \tau'^{-\frac{3}{2}} H(\tau'), \end{cases}$$

where $\qquad \tau = -\tau' = t - x/\alpha_1 - (2H-z)/\widehat{\beta_1\alpha_1} - h^2/2[x\alpha_1 - (2H-z)\,\widehat{\beta_1\alpha_1}];$

and similarly

$${}_p\psi^{(3)}_{\alpha_1} \doteqdot (-1)\times\ldots, \quad {}_pU^{(3)}_{\alpha_1},\, {}_p\dot{U}^{(3)}_{\alpha_1} \doteqdot (+1)\times\ldots, \quad {}_pW^{(3)}_{\alpha_1},\, {}_p\dot{W}^{(3)}_{\alpha_1} \doteqdot (-1)\times\ldots$$

with $\qquad (2H+z) \quad \text{for} \quad (2H-z),$

$${}_p\psi^{(8)}_{\alpha_1} \doteqdot (+1)\times\ldots, \quad {}_pU^{(8)}_{\alpha_1},\, {}_p\dot{U}^{(8)}_{\alpha_1} \doteqdot (-1)\times\ldots, \quad {}_pW^{(8)}_{\alpha_1},\, {}_p\dot{W}^{(8)}_{\alpha_1} \doteqdot (+1)\times\ldots$$

with $\qquad (2H-h) \text{ for } h, \quad z \text{ for } (2H-z),$

$${}_p\psi^{(9)}_{\alpha_1} \doteqdot (-1)\times\ldots, \quad {}_pU^{(9)}_{\alpha_1},\, {}_p\dot{U}^{(9)}_{\alpha_1} \doteqdot (+1)\times\ldots, \quad {}_pW^{(9)}_{\alpha_1},\, {}_p\dot{W}^{(9)}_{\alpha_1} \doteqdot (-1)\times\ldots$$

with $\qquad (2H+h) \text{ for } h, \quad z \text{ for } (2H-z);$

$$\begin{cases} {}_p\psi^{(4)}_{\alpha_1} \doteqdot -\sqrt{(2\pi i^3)}\,\alpha_1^2[x\alpha_1-(H-z)\,\widehat{\beta_1\alpha_1}]^{-\frac{3}{2}}\,(H-h)\,[Y_1/S_0]_{\alpha_1}\,\omega^{-\frac{1}{2}}\exp\{i\omega\tau\}, \\ \begin{cases} {}_pU^{(4)}_{\alpha_1} \doteqdot \sqrt{2}\,(\alpha_1^2/\widehat{\beta_1\alpha_1})\,[x\alpha_1-(H-z)\,\widehat{\beta_1\alpha_1}]^{-\frac{3}{2}}\,(H-h)\,[Y_1/S_0]_{\alpha_1,0}\,\tau^{-\frac{1}{2}}H(\tau), \\ {}_p\dot{U}^{(4)}_{\alpha_1} \doteqdot (\alpha_1^2/\sqrt{2}i\widehat{\beta_1\alpha_1})\,[x\alpha_1-(H-z)\,\widehat{\beta_1\alpha_1}]^{-\frac{3}{2}}\,(H-h)\,[Y_1/S_0]_{\alpha_1,1}\,\tau'^{-\frac{3}{2}}H(\tau'), \end{cases} \\ \begin{cases} {}_pW^{(4)}_{\alpha_1} \doteqdot \sqrt{2}\,\alpha_1[x\alpha_1-(H-z)\,\widehat{\beta_1\alpha_1}]^{-\frac{3}{2}}\,(H-h)\,[Y_1/S_0]_{\alpha_1,0}\,\tau^{-\frac{1}{2}}H(\tau), \\ {}_p\dot{W}^{(4)}_{\alpha_1} \doteqdot (\alpha_1/\sqrt{2}i)\,[x\alpha_1-(H-z)\,\widehat{\beta_1\alpha_1}]^{-\frac{3}{2}}\,(H-h)\,[Y_1/S_0]_{\alpha_1,1}\,\tau'^{-\frac{3}{2}}H(\tau'), \end{cases} \end{cases}$$

where $\tau = -\tau' = t-x/\alpha_1-(H-z)/\widehat{\beta_1\alpha_1}-\frac{1}{2}(H-h)^2/[x\alpha_1-(H-z)\,\widehat{\beta_1\alpha_1}];$

and similarly

$${}_p\psi^{(5)}_{\alpha_1} \doteqdot (-1)\times\ldots,\quad {}_pU^{(5)}_{\alpha_1},\,{}_p\dot{U}^{(5)}_{\alpha_1} \doteqdot (+1)\times\ldots,\quad {}_pW^{(5)}_{\alpha_1},\,{}_p\dot{W}^{(5)}_{\alpha_1} \doteqdot (-1)\times\ldots$$

with $(H+z)$ for $(H-z)$,

$${}_p\psi^{(6)}_{\alpha_1} \doteqdot (-1)\times\ldots,\quad {}_pU^{(6)}_{\alpha_1},\,{}_p\dot{U}^{(6)}_{\alpha_1} \doteqdot (-1)\times\ldots,\quad {}_pW^{(6)}_{\alpha_1},\,{}_p\dot{W}^{(6)}_{\alpha_1} \doteqdot (-1)\times\ldots$$

with $(H+h)$ for $(H-h)$,

$${}_p\psi^{(7)}_{\alpha_1} \doteqdot (+1)\times\ldots,\quad {}_pU^{(7)}_{\alpha_1},\,{}_p\dot{U}^{(7)}_{\alpha_1} \doteqdot (-1)\times\ldots,\quad {}_pW^{(7)}_{\alpha_1},\,{}_p\dot{W}^{(7)}_{\alpha_1} \doteqdot (+1)\times\ldots$$

with $(H+z)$ for $(H-z)$ and $(H+h)$ for $(H-h)$.

${}_p\phi$ contributions from Γ_{β_2}

$$\begin{cases} {}_p\phi^{(2)}_{\beta_2} \doteqdot \dfrac{-16\sqrt{(2\pi i^3)}\,(2/\beta_2^2-1/\beta_1^2)^2\beta_2^8}{\widehat{\beta_1\beta_2}(x\beta_2-2H\widehat{\beta_1\beta_2})^{\frac{3}{2}}}\left[\dfrac{1}{F_0^2}\left(\dfrac{T_1}{S_0}-\dfrac{T_0S_1}{S_0^2}\right)\right]_{\beta_2}\omega^{-\frac{3}{2}}\exp\{i\omega\tau-\omega p\}, \\ {}_p\dot{U}^{(2)}_{\beta_2} \doteqdot \dfrac{-16\sqrt{2}\,i\beta_2^7(2/\beta_2^2-1/\beta_1^2)^2}{\widehat{\beta_1\beta_2}(x\beta_2-2H\widehat{\beta_1\beta_2})^{\frac{3}{2}}}\left(\dfrac{h+z}{\widehat{\beta_2\alpha_1}}\right)^{-\frac{1}{2}}\begin{Bmatrix} i \\ 1\end{Bmatrix}\left[\dfrac{1}{F_0^2}\left(\dfrac{T_1}{S_0}-\dfrac{T_0S_1}{S_0^2}\right)\right]_{\beta_2,\,1,\,0,}\sin(\tfrac{1}{2}\psi\mp\tfrac{1}{4}\pi)\cos^{\frac{1}{2}}\psi, \\ {}_p\dot{W}^{(2)}_{\beta_2} \doteqdot \dfrac{-16\sqrt{2}\,i\beta_2^8(2/\beta_2^2-1/\beta_1^2)^2}{\widehat{\beta_1\beta_2}\,\widehat{\beta_2\alpha_1}(x\beta_2-2H\widehat{\beta_1\beta_2})^{\frac{3}{2}}}\left(\dfrac{h+z}{\widehat{\beta_2\alpha_1}}\right)^{-\frac{1}{2}}\begin{Bmatrix} 1 \\ -i\end{Bmatrix}\left[\dfrac{1}{F_0^2}\left(\dfrac{T_1}{S_0}-\dfrac{T_0S_1}{S_0^2}\right)\right]_{\beta_2,\,0,\,1,}\sin(\tfrac{1}{2}\psi\mp\tfrac{1}{4}\pi)\cos^{\frac{1}{2}}\psi, \end{cases}$$

where $\tau = t-x/\beta_2-2H/\widehat{\beta_1\beta_2},\quad p = (h+z)/\widehat{\beta_2\alpha_1},\quad \tan\psi = \tau/p;$

$$\begin{cases} {}_p\phi^{(3)}_{\beta_2} \doteqdot \dfrac{4\sqrt{(2\pi i^3)}\,\beta_2^5\,\widehat{\beta_2\alpha_1}(2/\beta_2^2-1/\beta_1^2)}{\widehat{\beta_1\beta_2}(x\beta_2-H\widehat{\beta_1\beta_2})^{\frac{3}{2}}}\left[\dfrac{1}{F_0}\left(\dfrac{Y_1}{S_0}-\dfrac{Y_0S_1}{S_0^2}\right)\right]_{\beta_2}\omega^{-\frac{3}{2}}\exp\{i\omega\tau-\omega p\}, \\ {}_p\dot{U}^{(3)}_{\beta_2} \doteqdot \dfrac{4\sqrt{2}\,i\,\widehat{\beta_2\alpha_1}\,\beta_2^4(2/\beta_2^2-1/\beta_1^2)}{\widehat{\beta_1\beta_2}(x\beta_2-H\widehat{\beta_1\beta_2})^{\frac{3}{2}}}\left(\dfrac{H+h-z}{\widehat{\beta_2\alpha_1}}\right)^{-\frac{1}{2}}\begin{Bmatrix} i \\ 1\end{Bmatrix}\left[\dfrac{1}{F_0}\left(\dfrac{Y_1}{S_0}-\dfrac{Y_0S_1}{S_0^2}\right)\right]_{\beta_2,\,0,\,1,}\sin(\tfrac{1}{2}\psi\mp\tfrac{1}{4}\pi)\cos^{\frac{1}{2}}\psi, \\ {}_p\dot{W}^{(3)}_{\beta_2} \doteqdot \dfrac{4\sqrt{2}\,i\beta_2^5(2/\beta_2^2-1/\beta_1^2)}{\widehat{\beta_1\beta_2}(x\beta_2-H\widehat{\beta_1\beta_2})^{\frac{3}{2}}}\left(\dfrac{H+h-z}{\widehat{\beta_2\alpha_1}}\right)\begin{Bmatrix} 1 \\ -i\end{Bmatrix}\left[\dfrac{1}{F_0}\left(\dfrac{Y_1}{S_0}-\dfrac{Y_0S_1}{S_0^2}\right)\right]_{\beta_2,\,1,\,0,}\sin(\tfrac{1}{2}\psi\mp\tfrac{1}{4}\pi)\cos^{\frac{1}{2}}\psi, \end{cases}$$

where $\tau = t-x/\beta_2-H/\widehat{\beta_1\beta_2},\quad p = (H+h-z)/\widehat{\beta_2\alpha_1},\quad \tan\psi = \tau/p;$

and similarly $\quad {}_p\phi^{(4)}_{\beta_2} \doteqdot (+1)\times\ldots,\quad {}_p\dot{U}^{(4)}_{\beta_2} \doteqdot (+1)\times\ldots,\quad {}_p\dot{W}^{(4)}_{\beta_2} \doteqdot (-1)\times\ldots$

with $(H-h+z)$ for $(H+h-z)$,

$${}_p\phi^{(5)}_{\beta_2} \doteqdot (-2)\times\ldots,\quad {}_p\dot{U}^{(5)}_{\beta_2} \doteqdot (-2)\times\ldots,\quad {}_p\dot{W}^{(5)}_{\beta_2} \doteqdot (+2)\times\ldots$$

with $(H+h+z)$ for $(H+h-z)$ and $[\overline{F}/F_0^2]$ for $[1/F_0]$;

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

$$\begin{cases} {}_p\phi^{(6)}_{\beta_2} \doteqdot -\surd(2\pi \mathrm{i})\, \beta_2^{\frac{1}{2}} \widehat{\beta_2\alpha_1}\, x^{-\frac{3}{2}} [W_1/S_0 - W_0 S_1/S_0^2]_{\beta_2}\, \omega^{-\frac{3}{2}} \exp\{\mathrm{i}\omega\tau - \omega p\}, \\ {}_p\dot{U}^{(6)}_{\beta_2} \doteqdot -\surd 2 \widehat{\beta_2\alpha_1}\, \beta_2^{-\frac{1}{2}} x^{-\frac{3}{2}} \left(\dfrac{2H-h-z}{\widehat{\beta_2\alpha_1}}\right)^{-\frac{1}{2}} \begin{Bmatrix} \mathrm{i} \\ 1 \end{Bmatrix} \left[\dfrac{W_1}{S_0} - \dfrac{W_0 S_1}{S_0^2}\right]_{\beta_2,\,\substack{0,\\1,}} \sin\left(\tfrac{1}{2}\psi \mp \tfrac{1}{4}\pi\right) \cos^{\frac{1}{2}}\psi, \\ {}_p W^{(6)}_{\beta_2} \doteqdot -\surd 2 \beta_2^{\frac{1}{2}} x^{-\frac{3}{2}} \left(\dfrac{2H-h-z}{\widehat{\beta_2\alpha_1}}\right)^{-\frac{1}{2}} \begin{Bmatrix} 1 \\ -\mathrm{i} \end{Bmatrix} \left[\dfrac{W_1}{S_0} - \dfrac{W_0 S_1}{S_0^2}\right]_{\beta_2,\,\substack{1,\\0,}} \sin\left(\tfrac{1}{2}\psi \mp \tfrac{1}{4}\pi\right) \cos^{\frac{1}{2}}\psi, \end{cases}$$

where $\tau = t - x/\beta_2, \quad p = (2H-h-z)/\widehat{\beta_2\alpha_1}, \quad \tan\psi = \tau/p,$

and similarly ${}_p\phi^{(7)}_{\beta_2} \doteqdot (-1) \times \ldots, \quad {}_p\dot{U}^{(7)}_{\beta_2} \doteqdot (-1) \times \ldots, \quad {}_p\dot{W}^{(7)}_{\beta_2} \doteqdot (-1) \times \ldots$
with

$(2H+h-z)$ for $(2H-h-z)$ and $[(W_1/S_0 - W_0 S_1/S_0^2)\, \overline{F}_0/F_0]$ for $[W_1/S_0 - W_0 S_1/S_0^2]$,

${}_p\phi^{(8)}_{\beta_2} \doteqdot (-1) \times \ldots, \quad {}_p\dot{U}^{(8)}_{\beta_2} \doteqdot (-1) \times \ldots, \quad {}_p\dot{W}^{(8)}_{\beta_2} \doteqdot (+1) \times \ldots$

with

$(2H-h+z)$ for $(2H-h-z)$ and $[(W_1/S_0 - W_0 S_1/S_0^2)\, \overline{F}_0/F_0]$ for $[W_1/S_0 - W_0 S_1/S_0^2]$,

${}_p\phi^{(9)}_{\beta_2} \doteqdot (+1) \times \ldots, \quad {}_p\dot{U}^{(9)}_{\beta_2} \doteqdot (+1) \times \ldots, \quad {}_p\dot{W}^{(9)}_{\beta_2} \doteqdot (-1) \times \ldots$

with

$(2H+h+z)$ for $(2H-h-z)$ and $[(W_1/S_0 - W_0 S_1/S_0^2)\, \overline{F}_0^2/F_0^2]$ for $[W_1/S_0 - W_0 S_1/S_0^2]$.

${}_p\psi$ contributions from Γ_{β_2}

$$\begin{cases} {}_p\psi^{(2)}_{\beta_2} \doteqdot \dfrac{-4\surd(2\pi \mathrm{i}^3)\, \beta_2^5 (2/\beta_2^2 - 1/\beta_1^2)}{[x\beta_2 - (2H-z)\, \widehat{\beta_1\beta_2}]^{\frac{3}{2}}} \left[\dfrac{1}{F_0}\left(\dfrac{T_1}{S_0} - \dfrac{T_0 S_1}{S_0^2}\right)\right]_{\beta_2} \omega^{-\frac{3}{2}} \exp\{\mathrm{i}\omega\tau - \omega p\}, \\ {}_p\dot{U}^{(2)}_{\beta_2} = \dfrac{4\surd 2\, \mathrm{i}\, (2/\beta_2^2 - 1/\beta_1^2)\, \beta_2^5}{\widehat{\beta_1\beta_2}[x\beta_2 - (2H-z)\, \widehat{\beta_1\beta_2}]^{\frac{3}{2}}} \left(\dfrac{h}{\widehat{\beta_2\alpha_1}}\right)^{-\frac{1}{2}} \begin{Bmatrix} \mathrm{i} \\ 1 \end{Bmatrix} \left[\dfrac{1}{F_0}\left(\dfrac{T_1}{S_0} - \dfrac{T_0 S_1}{S_0^2}\right)\right]_{\beta_2,\,\substack{1,\\0,}} \sin\left(\tfrac{1}{2}\psi \mp \tfrac{1}{4}\pi\right) \cos^{\frac{1}{2}}\psi, \\ {}_p\dot{W}^{(2)}_{\beta_2} = \dfrac{4\surd 2\, \mathrm{i}\, (2/\beta_2^2 - 1/\beta_1^2)\, \beta_2^4}{[x\beta_2 - (2H-z)\, \widehat{\beta_1\beta_2}]^{\frac{3}{2}}} \left(\dfrac{h}{\widehat{\beta_2\alpha_1}}\right)^{-\frac{1}{2}} \begin{Bmatrix} 1 \\ -\mathrm{i} \end{Bmatrix} \left[\dfrac{1}{F_0}\left(\dfrac{T_1}{S_0} - \dfrac{T_0 S_1}{S_0^2}\right)\right]_{\beta_2,\,\substack{0,\\1,}} \sin\left(\tfrac{1}{2}\psi \mp \tfrac{1}{4}\pi\right) \cos^{\frac{1}{2}}\psi, \end{cases}$$

where $\tau = t - x/\beta_2 - (2H-z)/\widehat{\beta_1\beta_2}, \quad p = h/\widehat{\beta_2\alpha_1}, \quad \tan\psi = \tau/p,$

and similarly ${}_p\psi^{(3)}_{\beta_2} \doteqdot (-1) \times \ldots, \quad {}_p\dot{U}^{(3)}_{\beta_2} \doteqdot (+1) \times \ldots, \quad {}_p\dot{W}^{(3)}_{\beta_2} \doteqdot (-1) \times \ldots$

with $(2H+z)$ for $(2H-z)$ and $(\overline{F}_0/F_0^2)$ for $(1/F_0)$;

$$\begin{cases} {}_p\psi^{(4)}_{\beta_2} \doteqdot \dfrac{\surd(2\pi \mathrm{i}^3)\, \beta_2^2 \widehat{\beta_2\alpha_1}}{[x\beta_2 - (H-z)\, \widehat{\beta_1\beta_2}]^{\frac{3}{2}}} \left[\dfrac{Y_1}{S_0} - \dfrac{Y_0 S_1}{S_0^2}\right]'_{\beta_2} \omega^{-\frac{3}{2}} \exp\{\mathrm{i}\omega\tau - \omega p\}, \\ {}_p\dot{U}^{(4)}_{\beta_2} \doteqdot \dfrac{-\surd 2\, \mathrm{i}\, \widehat{\beta_2\alpha_1}\, \beta_2^2}{\widehat{\beta_1\beta_2}[x\beta_2 - (H-z)\, \widehat{\beta_1\beta_2}]^{\frac{3}{2}}} \left(\dfrac{H-h}{\widehat{\beta_2\alpha_1}}\right)^{-\frac{1}{2}} \begin{Bmatrix} \mathrm{i} \\ 1 \end{Bmatrix} \left[\dfrac{Y_1}{S_0} - \dfrac{Y_0 S_1}{S_0^2}\right]_{\beta_2,\,\substack{0,\\1,}} \sin\left(\tfrac{1}{2}\psi \mp \tfrac{1}{4}\pi\right) \cos^{\frac{1}{2}}\psi, \\ {}_p\dot{W}^{(4)}_{\beta_2} \doteqdot \dfrac{-\surd 2\, \mathrm{i}\, \widehat{\beta_2\alpha_1}\, \beta_2}{[x\beta_2 - (H-z)\, \widehat{\beta_1\beta_2}]^{\frac{3}{2}}} \left(\dfrac{H-h}{\widehat{\beta_2\alpha_1}}\right)^{-\frac{1}{2}} \begin{Bmatrix} 1 \\ -\mathrm{i} \end{Bmatrix} \left[\dfrac{Y_1}{S_0} - \dfrac{Y_0 S_1}{S_0^2}\right]_{\beta_2,\,\substack{1,\\0,}} \sin\left(\tfrac{1}{2}\psi \mp \tfrac{1}{4}\pi\right) \cos^{\frac{1}{2}}\psi, \end{cases}$$

where $\tau = t - x/\beta_2 - (H-z)/\widehat{\beta_1\beta_2}, \quad p = (H-h)/\widehat{\beta_2\alpha_1}, \quad \tan\psi = \tau/p,$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

and similarly $\quad {}_p\psi^{(5)}_{\beta_2} \doteqdot (-1)\times\ldots, \quad {}_p\dot{U}^{(5)}_{\beta_2} \doteqdot (+1)\times\ldots, \quad {}_pW^{(5)}_{\beta_2} \doteqdot (-1)\times\ldots$

with $\quad (H+z)$ for $(H-z)$ and $[(Y_1/S_0-Y_0S_1/S_0^2)\,\overline{F}_0/F_0]$ for $[Y_1/S_0-Y_0S_1/S_0^2]$,

$$ {}_p\psi^{(6)}_{\beta_2} \doteqdot (-1)\times\ldots, \quad {}_p\dot{U}^{(6)}_{\beta_2} \doteqdot (-1)\times\ldots, \quad {}_p\dot{W}^{(6)}_{\beta_2} \doteqdot (-1)\times\ldots $$

with $\quad (H+h)$ for $(H-h)$ and $[(Y_1/S_0-Y_0S_1/S_0^2)\,\overline{F}_0/F_0]$ for $[Y_1/S_0-Y_0S_1/S_0^2]$,

$$ {}_p\psi^{(7)}_{\beta_2} \doteqdot (+1)\times\ldots, \quad {}_p\dot{U}^{(7)}_{\beta_2} \doteqdot (-1)\times\ldots, \quad {}_p\dot{W}^{(7)}_{\beta_2} \doteqdot (+1)\times\ldots $$

with $\quad (H+z)$ for $(H-z)$, $\quad (H+h)$ for $(H-h)$

and $[(\overline{F}_0^2/F_0^2+16\mathrm{i}\,\{2/\beta_2^2-1/\beta_1^2\}^2\,\beta_2^6/\widehat{\beta_2\alpha_1}\,\widehat{\beta_1\beta_2}\,F_0^2)\,(Y_1/S_0-Y_0S_1/S_0^2)]$ for $[Y_1/S_0-Y_0S_1/S_0^2]$;

$$ \begin{cases} {}_p\psi^{(8)}_{\beta_2} \doteqdot \dfrac{-4\sqrt{(2\pi\mathrm{i}^3)}\,\beta_2^5(2/\beta_2^2-1/\beta_1^2)}{(x\beta_2-z\widehat{\beta_1\beta_2})^{\frac{3}{2}}}\left[\dfrac{1}{F_0}\left(\dfrac{W_1}{S_0}-\dfrac{W_0S_1}{S_0^2}\right)\right]_{\beta_2}\omega^{-\frac{3}{2}}\exp\{\mathrm{i}\omega\tau-\omega p\}, \\ {}_p\dot{U}^{(8)}_{\beta_2} \doteqdot \dfrac{-4\sqrt{2}\,\mathrm{i}\,\beta_2^5(2/\beta_2^2-1/\beta_1^2)}{\widehat{\beta_1\beta_2}(x\beta_2-z\widehat{\beta_1\beta_2})^{\frac{3}{2}}}\left(\dfrac{2H-h}{\widehat{\beta_2\alpha_1}}\right)^{-\frac{1}{2}}\begin{cases}\mathrm{i}\\1\end{cases}\left[\dfrac{1}{F_0}\left(\dfrac{W_1}{S_0}-\dfrac{W_0S_1}{S_0^2}\right)\right]_{\beta_2,\,{1,\atop 0,}}\sin\left(\tfrac{1}{2}\psi\mp\tfrac{1}{4}\pi\right)\cos^{\frac{1}{2}}\psi, \\ {}_pW^{(8)}_{\beta_2} \doteqdot \dfrac{4\sqrt{2}\,\mathrm{i}\,\beta_2^4(2/\beta_2^2-1/\beta_1^2)}{(x\beta_2-z\widehat{\beta_1\beta_2})^{\frac{3}{2}}}\left(\dfrac{2H-h}{\widehat{\beta_2\alpha_1}}\right)^{-\frac{1}{2}}\begin{cases}1\\-\mathrm{i}\end{cases}\left[\dfrac{1}{F_0}\left(\dfrac{W_1}{S_0}-\dfrac{W_0S_1}{S_0^2}\right)\right]_{\beta_2,\,{0,\atop 1,}}\sin\left(\tfrac{1}{2}\psi\mp\tfrac{1}{4}\pi\right)\cos^{\frac{1}{2}}\psi, \end{cases} $$

where $\quad \tau = t-x/\beta_2-z/\widehat{\beta_1\beta_2}, \quad p=(2H-h)/\widehat{\beta_2\alpha_1}, \quad \tan\psi=\tau/p,$

and similarly $\quad {}_p\psi^{(9)}_{\beta_2} \doteqdot (-1)\times\ldots, \quad {}_p\dot{U}^{(9)}_{\beta_2} \doteqdot (-1)\times\ldots, \quad {}_p\dot{W}^{(9)}_{\beta_2} \doteqdot (-1)\times\ldots$

with $\quad (2H+h)$ for $(2H-h)$ and $[\overline{F}_0/F_0^2]$ for $[1/F_0]$.

${}_p\phi$ contributions from Γ_{β_1}

$$ \begin{cases} {}_p\phi^{(1)}_{\beta_1} \doteqdot -8\sqrt{(2\pi\mathrm{i})}\,\beta_1^{\frac{3}{2}}x^{-\frac{3}{2}}\omega^{-\frac{3}{2}}\exp\{\mathrm{i}\omega\tau-\omega p\}, \\ {}_p\dot{U}^{(1)}_{\beta_1} \doteqdot -8\sqrt{2}\,\beta_1^{\frac{1}{2}}x^{-\frac{3}{2}}\{(h+z)/\widehat{\beta_1\alpha_1}\}^{-\frac{1}{2}}\cos^{\frac{1}{2}}\psi\,\sin\left(\tfrac{1}{2}\psi+\tfrac{1}{4}\pi\right), \\ {}_p\dot{W}^{(1)}_{\beta_1} \doteqdot -8\sqrt{2}\,\beta_1^{\frac{3}{2}}\widehat{\beta_1\alpha_1}^{-1}x^{-\frac{3}{2}}\{(h+z)/\widehat{\beta_1\alpha_1}\}^{-\frac{1}{2}}\cos^{\frac{1}{2}}\psi\,\sin\left(\tfrac{1}{2}\psi-\tfrac{1}{4}\pi\right), \end{cases} $$

where $\quad \tau=t-x/\beta_1, \quad p=(h+z)/\widehat{\beta_1\alpha_1}, \quad \tan\psi=\tau/p;$

$$ \begin{cases} {}_p\phi^{(2)}_{\beta_1} \doteqdot -16\sqrt{(2\pi\mathrm{i})}\,\beta_1^{\frac{3}{2}}x^{-\frac{3}{2}}\omega^{-\frac{3}{2}}[T_0/S_0]_{\beta_1}\exp\{\mathrm{i}\omega\tau-\omega p\}, \\ {}_p\dot{U}^{(2)}_{\beta_1} \doteqdot -16\sqrt{2}\,\beta_1^{\frac{1}{2}}x^{-\frac{3}{2}}[T_0/S_0]_{\beta_1,0}\{(h+z)/\widehat{\beta_1\alpha_1}\}^{-\frac{1}{2}}\cos^{\frac{1}{2}}\psi\,\sin\left(\tfrac{1}{2}\psi+\tfrac{1}{4}\pi\right), \\ {}_p\dot{W}^{(2)}_{\beta_1} \doteqdot -16\sqrt{2}\,\beta_1^{\frac{3}{2}}\widehat{\beta_1\alpha_1}^{-1}x^{-\frac{3}{2}}[T_0/S_0]_{\beta_1,0}\{(h+z)/\widehat{\beta_1\alpha_1}\}^{-\frac{1}{2}}\cos^{\frac{1}{2}}\psi\,\sin\left(\tfrac{1}{2}\psi-\tfrac{1}{4}\pi\right), \end{cases} $$

where $\quad \tau=t-x/\beta_1-(2H)^2/2x\beta_1, \quad p=(h+z)/\widehat{\beta_1\alpha_1}, \quad \tan\psi=\tau/p;$

$$ \begin{cases} {}_p\phi^{(3)}_{\beta_1} \doteqdot 4\sqrt{(2\pi\mathrm{i})}\,\beta_1^{\frac{1}{2}}\widehat{\beta_1\alpha_1}\,x^{-\frac{3}{2}}\omega^{-\frac{3}{2}}[Y_0/S_0]_{\beta_1}\exp\{\mathrm{i}\omega\tau-\omega p\}, \\ {}_p\dot{U}^{(3)}_{\beta_1} \doteqdot 4\sqrt{2}\,\beta_1^{-\frac{1}{2}}\widehat{\beta_1\alpha_1}\,x^{-\frac{3}{2}}[Y_0/S_0]_{\beta_1,1}\{(H+h-z)/\widehat{\beta_1\alpha_1}\}^{-\frac{1}{2}}\cos^{\frac{1}{2}}\psi\,\sin\left(\tfrac{1}{2}\psi+\tfrac{1}{4}\pi\right), \\ {}_p\dot{W}^{(3)}_{\beta_1} \doteqdot 4\sqrt{2}\,\beta_1^{\frac{1}{2}}x^{-\frac{3}{2}}[Y_0/S_0]_{\beta_1,1}\{(H+h-z)/\widehat{\beta_1\alpha_1}\}^{-\frac{1}{2}}\cos^{\frac{1}{2}}\psi\,\sin\left(\tfrac{1}{2}\psi-\tfrac{1}{4}\pi\right), \end{cases} $$

where $\quad \tau=t-x/\beta_1-H^2/2x\beta_1, \quad p=(H+h-z)/\widehat{\beta_1\alpha_1}, \quad \tan\psi=\tau/p,$

and similarly $\;{}_p\phi^{(4)}_{\beta_1} \doteqdot (+1)\times\ldots,\quad {}_p\dot{U}^{(4)}_{\beta_1} \doteqdot (+1)\times\ldots,\quad {}_p\dot{W}^{(4)}_{\beta_1} \doteqdot (-1)\times\ldots$

with $\quad (H-h+z)\quad$ for $\quad (H+h-z)$,

$${}_p\phi^{(5)}_{\beta_1} \doteqdot (-2)\times\ldots,\quad {}_p\dot{U}^{(5)}_{\beta_1} \doteqdot (-2)\times\ldots,\quad {}_p W^{(5)}_{\beta_1} \doteqdot (+2)\times\ldots$$

with $\quad (H+h+z)\quad$ for $\quad (H+h-z)$,

$$\begin{cases} {}_p\phi^{(6)}_{\beta_1} \doteqdot -\sqrt{(2\pi\mathrm{i})}\,\beta_1^{\frac{1}{2}}\widehat{\beta_1\alpha_1}\,x^{-\frac{3}{2}}\omega^{-\frac{3}{2}}\left[\dfrac{W_1}{S_0}-\dfrac{W_0S_1}{S_0^2}\right]_{\beta_1}\exp\{\mathrm{i}\omega\tau-\omega p\},\\[2ex] {}_p\dot{U}^{(6)}_{\beta_1} \doteqdot -\sqrt{2}\,\widehat{\beta_1\alpha_1}\,\beta_1^{-\frac{1}{2}}x^{-\frac{3}{2}}\left[\dfrac{W_1}{S_0}-\dfrac{W_0S_1}{S_0^2}\right]_{\beta_1,1}\left(\dfrac{2H-h-z}{\widehat{\beta_1\alpha_1}}\right)^{-\frac{1}{2}}\cos^{\frac{1}{2}}\psi\,\sin\left(\tfrac{1}{2}\psi+\tfrac{1}{4}\pi\right),\\[2ex] {}_p\dot{W}^{(6)}_{\beta_1} \doteqdot -\sqrt{2}\,\beta_1^{\frac{1}{2}}x^{-\frac{3}{2}}\left[\dfrac{W_1}{S_0}-\dfrac{W_0S_1}{S_0^2}\right]_{\beta_1,1}\left(\dfrac{2H-h-z}{\widehat{\beta_1\alpha_1}}\right)^{-\frac{1}{2}}\cos^{\frac{1}{2}}\psi\,\sin\left(\tfrac{1}{2}\psi-\tfrac{1}{4}\pi\right),\end{cases}$$

where $\quad \tau = t-x/\beta_1,\quad p=(2H-h-z)/\widehat{\beta_1\alpha_1},\quad \tan\psi=\tau/p,$

and similarly $\;{}_p\phi^{(7)}_{\beta_1} \doteqdot (-1)\times\ldots,\quad {}_p\dot{U}^{(7)}_{\beta_1} \doteqdot (-1)\times\ldots,\quad {}_p\dot{W}^{(7)}_{\beta_1} \doteqdot (-1)\times\ldots$

with $\quad (2H+h-z)\quad$ for $\quad (2H-h-z)$,

$${}_p\phi^{(8)}_{\beta_1} \doteqdot (-1)\times\ldots,\quad {}_p\dot{U}^{(8)}_{\beta_1} \doteqdot (-1)\times\ldots,\quad {}_p\dot{W}^{(8)}_{\beta_1} \doteqdot (+1)\times\ldots$$

with $\quad (2H-h+z)\quad$ for $\quad (2H-h-z)$,

$${}_p\phi^{(9)}_{\beta_1} \doteqdot (+1)\times\ldots,\quad {}_p\dot{U}^{(9)}_{\beta_1} \doteqdot (+1)\times\ldots,\quad {}_p\dot{W}^{(9)}_{\beta_1} \doteqdot (-1)\times\ldots$$

with $\quad (2H+h+z)\quad$ for $\quad (2H-h-z)$.

${}_p\psi$ *contributions from* Γ_{β_1}

$$\begin{cases} {}_p\psi^{(1)}_{\beta_1} \doteqdot -4\sqrt{(2\pi\mathrm{i}^3)}\,\beta_1^{\frac{1}{2}}x^{-\frac{3}{2}}\{z\omega^{-\frac{1}{2}}+\beta_1\omega^{-\frac{3}{2}}[F_1/F_0]_{\beta_1}\}\exp\{\mathrm{i}\omega\tau-\omega p\},\\[1ex] {}_p\dot{U}^{(1)}_{\beta_1} \doteqdot 4\sqrt{2}\,\beta_1^{\frac{1}{2}}x^{-\frac{3}{2}}(h/\widehat{\beta_1\alpha_1})^{-\frac{1}{2}}\cos^{\frac{1}{2}}\psi\,\sin\left(\tfrac{1}{2}\psi+\tfrac{1}{4}\pi\right),\\[1ex] \begin{cases} {}_p W^{(1)}_{\beta_1} \doteqdot 4\sqrt{2}\,\beta_1^{-\frac{1}{2}}x^{-\frac{3}{2}}z(h/\widehat{\beta_1\alpha_1})^{-\frac{1}{2}}\cos^{\frac{1}{2}}\psi\,\sin\left(\tfrac{1}{2}\psi+\tfrac{1}{4}\pi\right),\\ {}_p\dot{W}^{(1)}_{\beta_1} \doteqdot 4\sqrt{2}\,\beta_1^{\frac{1}{2}}x^{-\frac{3}{2}}(h/\widehat{\beta_1\alpha_1})^{-\frac{1}{2}}[-F_1/F_0]_{\beta_1,1}\cos^{\frac{1}{2}}\psi\,\sin\left(\tfrac{1}{2}\psi-\tfrac{1}{4}\pi\right),\end{cases}\end{cases}$$

where $\quad \tau = t-x/\beta_1-z^2/2x\beta_1,\quad p=h/\widehat{\beta_1\alpha_1},\quad \tan\psi=\tau/p,$

and similarly

$${}_p\psi^{(2)}_{\beta_1} \doteqdot (-1)\times\ldots,\quad {}_p\dot{U}^{(2)}_{\beta_1} \doteqdot (+1)\times\ldots,\quad {}_p W^{(2)}_{\beta_1},\ {}_p\dot{W}^{(2)}_{\beta_1} \doteqdot (-1)\times\ldots$$

with $\quad (2H-z)$ for z, $\quad [2S_1/S_0+F_1/F_0]_{\beta_1}$ for $[F_1/F_0]_{\beta_1}$,

$${}_p\psi^{(3)}_{\beta_1} \doteqdot (+1)\times\ldots,\quad {}_p\dot{U}^{(3)}_{\beta_1} \doteqdot (+1)\times\ldots,\quad {}_p W^{(3)}_{\beta_1},\ {}_p\dot{W}^{(3)}_{\beta_1} \doteqdot (-1)\times\ldots$$

with $\quad (2H+z)$ for z, $\quad [2S_1/S_0+3F_1/F_0]$ for $[F_1/F_0]_{\beta_1}$;

$$\begin{cases} {}_p\psi^{(4)}_{\beta_1} \doteqdot -\sqrt{(2\pi\mathrm{i}^3)}\,\widehat{\beta_1\alpha_1}\,\beta_1^{-\frac{1}{2}}x^{-\frac{3}{2}}\{(H-z)\,\omega^{-\frac{1}{2}}[Y_0/S_0]_{\beta_1}+\beta_1\omega^{-\frac{3}{2}}[Y_0S_1/S_0^2]\}\exp\{\mathrm{i}\omega\tau-\omega p\},\\[1ex] {}_p\dot{U}^{(4)}_{\beta_1} \doteqdot -\sqrt{2}\,\widehat{\beta_1\alpha_1}\,\beta_1^{-\frac{1}{2}}x^{-\frac{3}{2}}[(H-h)/\widehat{\beta_1\alpha_1}]^{-\frac{1}{2}}[Y_0/S_0]_{\beta_1,1}\cos^{\frac{1}{2}}\psi\,\sin\left(\tfrac{1}{2}\psi+\tfrac{1}{4}\pi\right),\\[1ex] \begin{cases} {}_p W^{(4)}_{\beta_1} \doteqdot \sqrt{2}\,\widehat{\beta_1\alpha_1}\,\beta_1^{-\frac{3}{2}}x^{-\frac{3}{2}}(H-z)\,[(H-h)/\widehat{\beta_1\alpha_1}]^{-\frac{1}{2}}[Y_0/S_0]_{\beta_1,1}\cos^{\frac{1}{2}}\psi\,\sin\left(\tfrac{1}{2}\psi+\tfrac{1}{4}\pi\right),\\ {}_p\dot{W}^{(4)}_{\beta_1} \doteqdot -\sqrt{2}\,\widehat{\beta_1\alpha_1}\,\beta_1^{-\frac{1}{2}}x^{-\frac{3}{2}}[(H-h)/\widehat{\beta_1\alpha_1}]^{-\frac{1}{2}}[Y_0S_1/S_0^2]_{\beta_1,0}\cos^{\frac{1}{2}}\psi\,\sin\left(\tfrac{1}{2}\psi-\tfrac{1}{4}\pi\right),\end{cases}\end{cases}$$

where $\quad \tau = t-x/\beta_1-(H-z)^2/2x\beta_1,\quad p=(H-h)/\widehat{\beta_1\alpha_1},\quad \tan\psi=\tau/p,$

and similarly

$${}_p\psi^{(5)}_{\beta_1} \doteqdot (-1)\times\ldots, \quad {}_p\dot{U}^{(5)}_{\beta_1} \doteqdot (+1)\times\ldots, \quad {}_pW^{(5)}_{\beta_1},\ {}_p\dot{W}^{(5)}_{\beta_1} \doteqdot (-1)\times\ldots$$

with $(H+z)$ for $(H-z)$ and $[Y_0S_1/S_0^2+2Y_0F_1/S_0F_0]$ for $[Y_0S_1/S_0^2]$,

$${}_p\psi^{(6)}_{\beta_1} \doteqdot (-1)\times\ldots, \quad {}_p\dot{U}^{(6)}_{\beta_1} \doteqdot (-1)\times\ldots, \quad {}_pW^{(6)}_{\beta_1},\ {}_p\dot{W}^{(6)}_{\beta_1} \doteqdot (-1)\times\ldots$$

with $(H+h)$ for $(H-h)$ and $[Y_0S_1/S_0^2+2Y_0F_1/S_0F_0]$ for $[Y_0S_1/S_0^2]$,

$${}_p\psi^{(7)}_{\beta_1} \doteqdot (+1)\times\ldots, \quad {}_p\dot{U}^{(7)}_{\beta_1} \doteqdot (-1)\times\ldots, \quad {}_pW^{(7)}_{\beta_1},\ {}_p\dot{W}^{(7)}_{\beta_1} \doteqdot (+1)\times\ldots$$

with $(H+z)$ for $(H-z)$, $(H+h)$ for $(H-h)$

and $[Y_0S_1/S_0^2+(4F_1/F_0-16\beta_1/\widehat{\beta_1\alpha_1})\,Y_0/S_0]$ for $[Y_0S_1/S_0^2]$;

$$\begin{cases} {}_p\psi^{(8)}_{\beta_1} \doteqdot 4\sqrt{(2\pi i^3)}\,\beta_1^{\frac{1}{2}}x^{-\frac{3}{2}}\left\{z\omega^{-\frac{1}{2}}\left[\dfrac{W_0}{S_0}\right]_{\beta_1} - \beta_1\omega^{-\frac{3}{2}}\left[\dfrac{W_1}{S_0}-\dfrac{W_0S_1}{S_0^2}-\dfrac{F_1}{F_0}\dfrac{W_0}{S_0}\right]_{\beta_1}\right\}\exp\{i\omega\tau-\omega p\}, \\ {}_p\dot{U}^{(8)}_{\beta_1} \doteqdot -4\sqrt{2}\,\beta_1^{\frac{1}{2}}x^{-\frac{3}{2}}[(2H-h)/\widehat{\beta_1\alpha_1}]^{-\frac{1}{2}}\left[\dfrac{W_0}{S_0}\right]_{\beta_1,0}\cos^{\frac{1}{2}}\psi\,\sin(\tfrac{1}{2}\psi+\tfrac{1}{4}\pi), \\ {}_pW^{(8)}_{\beta_1} \doteqdot -4\sqrt{2}\,\beta_1^{-\frac{1}{2}}x^{-\frac{3}{2}}z[(2H-h)/\widehat{\beta_1\alpha_1}]^{-\frac{1}{2}}\left[\dfrac{W_0}{S_0}\right]_{\beta_1,0}\cos^{\frac{1}{2}}\psi\,\sin(\tfrac{1}{2}\psi+\tfrac{1}{4}\pi), \\ {}_p\dot{W}^{(8)}_{\beta_1} \doteqdot -4\sqrt{2}\,\beta_1^{\frac{1}{2}}x^{-\frac{3}{2}}[(2H-h)/\widehat{\beta_1\alpha_1}]^{-\frac{1}{2}}\left[\dfrac{W_1}{S_0}-\dfrac{W_0S_1}{S_0^2}-\dfrac{F_1}{F_0}\dfrac{W_0}{S_0}\right]_{\beta_1,1}\cos^{\frac{1}{2}}\psi\,\sin(\tfrac{1}{2}\psi-\tfrac{1}{4}\pi), \end{cases}$$

where $\tau = t-x/\beta_1-z^2/2x\beta_1$, $p=(2H-h)/\widehat{\beta_1\alpha_1}$, $\tan\psi=\tau/p$,
and similarly

$${}_p\psi^{(9)}_{\beta_1} \doteqdot (-1)\times\ldots, \quad {}_p\dot{U}^{(9)}_{\beta_1} \doteqdot (-1)\times\ldots, \quad {}_pW^{(9)}_{\beta_1},\ {}_p\dot{W}^{(9)}_{\beta_1} \doteqdot (-1)\times\ldots$$

with $(2H+h)$ for $(2H-h)$ and $(3F_1W_0/F_0S_0)$ for (F_1W_0/F_0S_0).

${}_s\phi$ contributions from Γ_{α_2}

$$\begin{cases} {}_s\phi^{(2)}_{\alpha_2} \doteqdot \dfrac{4\sqrt{(2\pi i^3)}\,\alpha_2^5(2/\alpha_2^2-1/\beta_1^2)}{[x\alpha_2-z\widehat{\alpha_1\alpha_2}-(2H-h)\,\widehat{\beta_1\alpha_2}]^{\frac{3}{2}}}\left[\dfrac{T_1}{S_0}-\dfrac{T_0S_1}{S_0^2}\right]_{\alpha_2}\omega^{-\frac{3}{2}}\exp\{i\omega\tau\}, \\ {}_sU^{(2)}_{\alpha_2} \doteqdot \dfrac{8\sqrt{2}\,i\,\alpha_2^4(2/\alpha_2^2-1/\beta_1^2)}{[x\alpha_2-z\widehat{\alpha_1\alpha_2}-(2H-h)\,\widehat{\beta_1\alpha_2}]^{\frac{3}{2}}}\left[\dfrac{T_1}{S_0}-\dfrac{T_0S_1}{S_0^2}\right]_{\alpha_2,0}\tau^{\frac{1}{2}}H(\tau) \doteqdot {}_sW^{(2)}_{\alpha_2}\times\widehat{\alpha_1\alpha_2}/\alpha_2, \end{cases}$$

where $\tau = t-x/\alpha_2-z/\widehat{\alpha_1\alpha_2}-(2H-h)/\widehat{\beta_1\alpha_2}$;

and similarly

$${}_s\phi^{(2)}_{\alpha_2} \doteqdot (-1)\times\ldots, \quad {}_sU^{(3)}_{\alpha_2} \doteqdot (-1)\times\ldots, \quad {}_sW^{(3)}_{\alpha_2} \doteqdot (-1)\times\ldots,$$

with $(2H+h)$ for $(2H-h)$ and $[(T_1/S_0-T_0S_1/S_0^2)\,\overline{F}_0/F_0]$ for $[T_1/S_0-T_0S_1/S_0^2]$;

$$\begin{cases} {}_s\phi^{(4)}_{\alpha_2} \doteqdot \dfrac{-\sqrt{(2\pi i)}\,\widehat{\alpha_1\alpha_2}\,\alpha_2^2}{[x\alpha_2-(H-z)\,\widehat{\alpha_1\alpha_2}-(H-h)\,\widehat{\beta_1\alpha_2}]^{\frac{3}{2}}}\left[\dfrac{Y_1}{S_0}-\dfrac{Y_0S_1}{S_0^2}\right]_{\alpha_2}\omega^{-\frac{3}{2}}\exp\{i\omega\tau\}, \\ {}_sU^{(4)}_{\alpha_2} \doteqdot \dfrac{-2\sqrt{2}\,\widehat{\alpha_1\alpha_2}\,\alpha_2}{[x\alpha_2-(H-z)\,\widehat{\alpha_1\alpha_2}-(H-h)\,\widehat{\beta_1\alpha_2}]^{\frac{3}{2}}}\left[\dfrac{Y_1}{S_0}-\dfrac{Y_0S_1}{S_0^2}\right]_{\alpha_2,0}\tau^{\frac{1}{2}}H(\tau) \doteqdot -{}_sW^{(4)}_{\alpha_2}\times\widehat{\alpha_1\alpha_2}/\alpha_2, \end{cases}$$

where $\tau = t-x/\alpha_2-(H-z)/\widehat{\alpha_1\alpha_2}-(H-h)/\widehat{\beta_1\alpha_2}$;

and similarly

$$ {}_s\phi^{(5)}_{\alpha_2} \doteqdot (-1)\times\ldots, \quad {}_sU^{(5)}_{\alpha_2} \doteqdot (-1)\times\ldots, \quad {}_sW^{(5)}_{\alpha_2} \doteqdot (-1)\times\ldots, $$

with $(H+h)$ for $(H-h)$ and $[(Y_1/S_0-Y_0S_1/S_0^2)\,\overline{F}_0/F_0]$ for $[Y_1/S_0-Y_0S_1/S_0^2]$,

$$ {}_s\phi^{(6)}_{\alpha_2} \doteqdot (-1)\times\ldots, \quad {}_sU^{(6)}_{\alpha_2} \doteqdot (-1)\times\ldots, \quad {}_sW^{(6)}_{\alpha_2} \doteqdot (+1)\times\ldots, $$

with $(H+z)$ for $(H-z)$ and $[(Y_1/S_0-Y_0S_1/S_0^2)\,\overline{F}_0/F_0]$ for $[Y_1/S_0-Y_0S_1/S_0^2]$,

$$ {}_s\phi^{(7)}_{\alpha_2} \doteqdot (+1)\times\ldots, \quad {}_sU^{(7)}_{\alpha_2} \doteqdot (+1)\times\ldots, \quad {}_sW^{(7)}_{\alpha_2} \doteqdot (-1)\times\ldots, $$

with $(H+h)$ for $(H-h)$, $(H+z)$ for $(H-z)$

and

$$ [(Y_1/S_0-Y_0S_1/S_0^2)\,(\overline{F}{}_0^2/F_0^2-16\{2/\alpha_2^2-1/\beta_1^2\}^2\,\alpha_2^6/\widehat{\alpha_1\alpha_2}\,\widehat{\beta_1\alpha_2}\,F_0^2)] \text{ for } [Y_1/S_0-Y_0S_1/S_0^2]; $$

$$ \begin{cases} {}_s\phi^{(8)}_{\alpha_2} \doteqdot \dfrac{4\sqrt{(2\pi \mathrm{i}^3)}\,(2/\alpha_2^2-1/\beta_1^2)\,\alpha_2^5}{[x\alpha_2-(2H-z)\,\widehat{\alpha_1\alpha_2}-h\widehat{\beta_1\alpha_2}]^{\frac{1}{2}}}\left[\dfrac{1}{F_0}\left(\dfrac{W_1}{S_0}-\dfrac{W_0S_1}{S_0^2}\right)\right]_{\alpha_2}\omega^{-\frac{1}{2}}\exp\{\mathrm{i}\omega\tau\}, \\ {}_sU^{(8)}_{\alpha_2} \doteqdot \dfrac{8\sqrt{2}\,\mathrm{i}\,(2/\alpha_2^2-1/\beta_1^2)\,\alpha_2^4}{[x\alpha_2-(2H-z)\,\widehat{\alpha_1\alpha_2}-h\widehat{\beta_1\alpha_2}]^{\frac{1}{2}}}\left[\dfrac{1}{F_0}\left(\dfrac{W_1}{S_0}-\dfrac{W_0S_1}{S_0^2}\right)\right]_{\alpha_2,0}\tau^{\frac{1}{2}}H(\tau) \doteqdot -{}_sW^{(8)}_{\alpha_2}\times\widehat{\alpha_1\alpha_2}/\alpha_2, \end{cases} $$

where
$$ \tau = t-x/\alpha_2-(2H-z)/\widehat{\alpha_1\alpha_2}-h/\widehat{\beta_1\alpha_2}; $$

and similarly

$$ {}_s\phi^{(9)}_{\alpha_2} \doteqdot (-1)\times\ldots, \quad {}_sU^{(9)}_{\alpha_2} \doteqdot (-1)\times\ldots, \quad {}_sW^{(9)}_{\alpha_2} \doteqdot (+1)\times\ldots, $$

with $(2H+z)$ for $(2H-z)$ and $[\overline{F}_0/F_0^2]$ for $[1/\overline{F}_0]$.

${}_s\psi$ contributions from Γ_{α_2}

$$ \begin{cases} {}_s\psi^{(2)}_{\alpha_2} \doteqdot \dfrac{-16\sqrt{(2\pi \mathrm{i}^3)}\,\alpha_2^8(2/\alpha_2^2-1/\beta_1^2)^2}{\widehat{\alpha_1\alpha_2}[x\alpha_2-2H\widehat{\alpha_1\alpha_2}-(h+z)\,\widehat{\beta_1\alpha_2}]^{\frac{1}{2}}}\left[\dfrac{1}{F_0}\left(\dfrac{W_1}{S_0}-\dfrac{W_0S_1}{S_0^2}\right)\right]_{\alpha_2}\omega^{-\frac{1}{2}}\exp\{\mathrm{i}\omega\tau\}, \\ {}_sU^{(2)}_{\alpha_2} \doteqdot \dfrac{-32\sqrt{2}\,\mathrm{i}\,\alpha_2^8(2/\alpha_2^2-1/\beta_1^2)^2}{\widehat{\alpha_1\alpha_2}\,\widehat{\beta_1\alpha_2}[x\alpha_2-2H\widehat{\alpha_1\alpha_2}-(h+z)\,\widehat{\beta_1\alpha_2}]^{\frac{1}{2}}}\left[\dfrac{1}{F_0}\left(\dfrac{W_1}{S_0}-\dfrac{W_0S_1}{S_0^2}\right)\right]_{\alpha_2,0}\tau^{\frac{1}{2}}H(\tau) \doteqdot -{}_sW^{(2)}_{\alpha_2}\times\widehat{\alpha_1\alpha_2}/\alpha_2, \end{cases} $$

where
$$ \tau = t-x/\alpha_2-2H/\widehat{\alpha_1\alpha_2}-(h+z)/\widehat{\beta_1\alpha_2}, $$

$$ \begin{cases} {}_s\psi^{(3)}_{\alpha_2} \doteqdot \dfrac{4\sqrt{(2\pi \mathrm{i})}\,\alpha_2^5(2/\alpha_2^2-1/\beta_1^2)}{[x\alpha_2-H\widehat{\alpha_1\alpha_2}-(H+h-z)\,\widehat{\beta_1\alpha_2}]^{\frac{1}{2}}}\left[\dfrac{1}{F_0}\left(\dfrac{Y_1}{S_0}-\dfrac{Y_0S_1}{S_0^2}\right)\right]_{\alpha_2}\omega^{-\frac{1}{2}}\exp\{\mathrm{i}\omega\tau\}, \\ {}_sU^{(3)}_{\alpha_2} \doteqdot \dfrac{-8\sqrt{2}\,\alpha_2^5(2/\alpha_2^2-1/\beta_1^2)}{\widehat{\beta_1\alpha_2}[x\alpha_2-H\widehat{\alpha_1\alpha_2}-(H+h-z)\,\widehat{\beta_1\alpha_2}]^{\frac{1}{2}}}\left[\dfrac{1}{F_0}\left(\dfrac{Y_1}{S_0}-\dfrac{Y_0S_1}{S_0^2}\right)\right]_{\alpha_2,0}\tau^{\frac{1}{2}}H(\tau) = {}_sW^{(3)}_{\alpha_2}\times\widehat{\alpha_1\alpha_2}/\alpha_2, \end{cases} $$

where
$$ \tau = t-x/\alpha_2-H/\widehat{\alpha_1\alpha_2}-(H+h-z)/\widehat{\beta_1\alpha_2}, $$

and similarly

$$ {}_s\psi^{(4)}_{\alpha_2} \doteqdot (+1)\times\ldots, \quad {}_sU^{(4)}_{\alpha_2} \doteqdot (-1)\times\ldots, \quad {}_sW^{(4)}_{\alpha_2} \doteqdot (+1)\times\ldots, $$

with $(H-h+z)$ for $(H+h-z)$,

$$ {}_s\psi^{(5)}_{\alpha_2} \doteqdot (-2)\times\ldots, \quad {}_sU^{(5)}_{\alpha_2} \doteqdot (+2)\times\ldots, \quad {}_sW^{(5)}_{\alpha_2} \doteqdot (-2)\times\ldots, $$

with $(H+h+z)$ for $(H+h-z)$ and $[\overline{F}_0/F_0^2]$ for $[1/F_0]$;

$$\begin{cases} {}_s\psi^{(6)}_{\alpha_2} \doteqdot \dfrac{\sqrt{(2\pi i^3)}\,\alpha_2^2\widehat{\beta_1\alpha_2}}{[x\alpha_2-(2H-h-z)\,\widehat{\beta_1\alpha_2}]^{\frac{3}{2}}}\left[\dfrac{T_1}{S_0}-\dfrac{T_0S_1}{S_0^2}\right]_{\alpha_2}\omega^{-\frac{3}{2}}\exp\{i\omega\tau\}, \\ {}_sU^{(6)}_{\alpha_2} \doteqdot \dfrac{-2\sqrt{2}\,i\,\alpha_2^2}{[x\alpha_2-(2H-h-z)\,\widehat{\beta_1\alpha_2}]^{\frac{3}{2}}}\left[\dfrac{T_1}{S_0}-\dfrac{T_0S_1}{S_0^2}\right]_{\alpha_2,0}\tau^{\frac{1}{2}}H(\tau) \doteqdot {}_sW^{(6)}_{\alpha_2}\times\alpha_2/\widehat{\beta_1\alpha_2}, \end{cases}$$

where $$\tau = t-x/\alpha_2-(2H-h-z)/\widehat{\beta_1\alpha_2},$$

and similarly

$${}_s\psi^{(7)}_{\alpha_2}\doteqdot(-1)\times\ldots,\quad {}_sU^{(7)}_{\alpha_2}\doteqdot(-1)\times\ldots,\quad {}_sW^{(7)}_{\alpha_2}\doteqdot(-1)\times\ldots,$$

with $$(2H+h-z)\quad\text{for}\quad(2H-h-z)$$

and $$[(T_1/S_0-T_0S_1/S_0^2)\,\overline{F}_0/F_0]\quad\text{for}\quad[T_1/S_0-T_0S_1/S_0^2],$$

$${}_s\psi^{(8)}_{\alpha_2}\doteqdot(-1)\times\ldots,\quad {}_sU^{(8)}_{\alpha_2}\doteqdot(+1)\times\ldots,\quad {}_sW^{(8)}_{\alpha_2}\doteqdot(-1)\times\ldots,$$

with $$(2H-h+z)\quad\text{for}\quad(2H-h-z)$$

and $$[(T_1/S_0-T_0S_1/S_0^2)\,\overline{F}_0/F_0]\quad\text{for}\quad[T_1/S_0-T_0S_1/S_0^2],$$

$${}_s\psi^{(9)}_{\alpha_2}\doteqdot(+1)\times\ldots,\quad {}_sU^{(9)}_{\alpha_2}\doteqdot(-1)\times\ldots,\quad {}_sW^{(9)}_{\alpha_2}\doteqdot(+1)\times\ldots,$$

with $$(2H+h+z)\quad\text{for}\quad(2H-h-z)$$

and $$[(T_1/S_0-T_0S_1/S_0^2)\,\overline{F}_0^2/F_0^2]\quad\text{for}\quad[T_1/S_0-T_0S_1/S_0^2].$$

${}_s\phi$ contributions from Γ_{α_1}

$$\begin{cases} {}_s\phi^{(1)}_{\alpha_1} \doteqdot 4\sqrt{(2\pi i^3)}\,(2/\alpha_1^2-1/\beta_1^2)^{-1}\,(x\alpha_1-h\widehat{\beta_1\alpha_1})^{-\frac{3}{2}}\{z\omega^{-\frac{1}{2}}+\alpha_1[F_1/F_0]_{\alpha_1}\omega^{-\frac{3}{2}}\}\exp\{i\omega\tau\}, \\ {}_sU^{(1)}_{\alpha_1} \doteqdot \dfrac{4\sqrt{2}\,(x\alpha_1-H\widehat{\beta_1\alpha_1})^{-\frac{3}{2}}}{\alpha_1(2/\alpha_1^2-1/\beta_1^2)}\{z\tau^{-\frac{1}{2}}H(\tau)+2i\alpha_1[F_1/F_0]_{\alpha,0}\tau^{\frac{1}{2}}H(\tau)\}, \\ {}_sW^{(1)}_{\alpha_1} \doteqdot \dfrac{-8\sqrt{2}\,(x\alpha_1-h\widehat{\beta_1\alpha_1})^{-\frac{3}{2}}}{(2/\alpha_1^2-1/\beta_1^2)}\tau^{\frac{1}{2}}H(\tau), \end{cases}$$

where $$\tau = t-x/\alpha_1-h/\widehat{\beta_1\alpha_1}-z^2/2(x\alpha_1-h\widehat{\beta_1\alpha_1});$$

$$\begin{cases} {}_s\phi^{(2)}_{\alpha_1} \doteqdot -4\sqrt{(2\pi i^3)}\,(2/\alpha_1^2-1/\beta_1^2)^{-1}\,[x\alpha_1-(2H-h)\,\widehat{\beta_1\alpha_1}]^{-\frac{3}{2}}\exp\{i\omega\tau\} \\ \qquad\times\{z[T_0/S_0]_{\alpha_1}\omega^{-\frac{1}{2}}+\alpha_1[-T_1/S_0+T_0S_1/S_0^2+T_0F_1/S_0F_0]\,\omega^{-\frac{3}{2}}\}, \\ \begin{cases} {}_sU^{(2)}_{\alpha_1} \doteqdot \dfrac{-4\sqrt{2}\,[x\alpha_1-(2H-h)\,\widehat{\beta_1\alpha_1}]^{-\frac{3}{2}}}{\alpha_1(2/\alpha_1^2-1/\beta_1^2)} \\ \qquad\times\{z[T_0/S_0]_{\alpha_1,0}\tau^{-\frac{1}{2}}H(\tau)+\alpha_1[-T_1/S_0+T_0S_1/S_0^2+T_0F_1/S_0F_0]\,\tau^{\frac{1}{2}}H(\tau)\}, \\ {}_s\dot{U}^{(2)}_{\alpha_1} \doteqdot \dfrac{2\sqrt{2}\,i\,[x\alpha_1-(2H-h)\,\widehat{\beta_1\alpha_1}]^{-\frac{3}{2}}}{\alpha_1(2/\alpha_1^2-1/\beta_1^2)} \\ \qquad\times\{z[T_0/S_0]_{\alpha_1,1}\tau'^{-\frac{3}{2}}H(\tau')-2i\alpha_1[-T_1/S_0+T_0S_1/S_0^2+T_0F_1/S_0F_0]\,\tau'^{-\frac{1}{2}}H(\tau')\}, \end{cases} \\ \begin{cases} {}_sW^{(2)}_{\alpha_1} \doteqdot \dfrac{8\sqrt{2}\,[x\alpha_1-(2H-h)\,\widehat{\beta_1\alpha_1}]^{-\frac{3}{2}}}{(2/\alpha_1^2-1/\beta_1^2)}\left[\dfrac{T_0}{S_0}\right]_{\alpha_1,0}\tau^{\frac{1}{2}}H(\tau), \\ {}_s\dot{W}^{(2)}_{\alpha_1} \doteqdot \dfrac{-4\sqrt{2}\,i\,[x\alpha_1-(2H-h)\,\widehat{\beta_1\alpha_1}]^{-\frac{3}{2}}}{(2/\alpha_1^2-1/\beta_1^2)}\left[\dfrac{T_0}{S_0}\right]_{\alpha_1,1}\tau'^{-\frac{1}{2}}H(\tau'), \end{cases} \end{cases}$$

where $$\tau = t - x/\alpha_1 - (2H-h)/\widehat{\beta_1\alpha_1} - z^2/2[x\alpha_1 - (2H-h)/\widehat{\beta_1\alpha_1}],$$

and similarly

$${}_s\phi^{(3)}_{\alpha_1} \doteqdot (-1)\times\ldots,\quad {}_sU^{(3)}_{\alpha_1},\ {}_s\dot{U}^{(3)}_{\alpha_1} \doteqdot (-1)\times\ldots,\quad {}_sW^{(3)}_{\alpha_1},\ {}_s\dot{W}^{(3)}_{\alpha_1} \doteqdot (-1)\times\ldots$$

with $$(2H+h) \quad \text{for} \quad (2H-h);$$

$$\begin{cases} {}_s\phi^{(4)}_{\alpha_1} \doteqdot \surd(2\pi i^3)\,\alpha_1^2[x\alpha_1 - (H-h)\,\widehat{\beta_1\alpha_1}]^{-\frac{3}{2}}\,(H-z)\,[Y_1/S_0]_{\alpha_1}\,\omega^{-\frac{1}{2}}\exp\{i\omega\tau\}, \\ \begin{cases} {}_sU^{(4)}_{\alpha_1} \doteqdot \dfrac{\surd 2\,\alpha_1(H-z)}{[x\alpha_1 - (H-h)\,\widehat{\beta_1\alpha_1}]^{\frac{3}{2}}}\left[\dfrac{Y_1}{S_0}\right]_{\alpha_1,0}\tau^{-\frac{1}{2}}H(\tau), \\ {}_s\dot{U}^{(4)}_{\alpha_1} \doteqdot \dfrac{-\frac{1}{2}\surd 2\,i\,(H-z)}{-[x\alpha_1 - (H-h)\,\widehat{\beta_1\alpha_1}]^{\frac{3}{2}}}\left[\dfrac{Y_1}{S_0}\right]_{\alpha_1,1}\tau'^{-\frac{3}{2}}H(\tau'), \end{cases} \\ \begin{cases} {}_sW^{(4)}_{\alpha_1} \doteqdot 2\surd 2\,\alpha_1^2[x\alpha_1 - (H-h)\,\widehat{\beta_1\alpha_1}]^{-\frac{3}{2}}\,[Y_1/S_0]_{\alpha_1,0}\,\tau^{\frac{1}{2}}H(\tau), \\ {}_s\dot{W}^{(4)}_{\alpha_1} \doteqdot -\surd 2\,i\,\alpha_1^2[x\alpha_1 - (H-h)\,\widehat{\beta_1\alpha_1}]^{-\frac{3}{2}}\,[Y_1/S_0]_{\alpha_1,1}\,\tau'^{-\frac{1}{2}}H(\tau'), \end{cases} \end{cases}$$

where $$\tau = -\tau' = t - x/\alpha_1 - (H-h)/\widehat{\beta_1\alpha_1} - (H-z)^2/2[x\alpha_1 - (H-h)\,\widehat{\beta_1\alpha_1}],$$

and similarly

$${}_s\phi^{(5)}_{\alpha_1} \doteqdot (-1)\times\ldots,\quad {}_sU^{(5)}_{\alpha_1},\ {}_s\dot{U}^{(5)}_{\alpha_1} \doteqdot (-1)\times\ldots,\quad {}_sW^{(5)}_{\alpha_1},\ {}_s\dot{W}^{(5)}_{\alpha_1} \doteqdot (-1)\times\ldots$$

with $$(H+h) \quad \text{for} \quad (H-h),$$

$${}_s\phi^{(6)}_{\alpha_1} \doteqdot (-1)\times\ldots,\quad {}_sU^{(6)}_{\alpha_1},\ {}_s\dot{U}^{(6)}_{\alpha_1} \doteqdot (-1)\times\ldots,\quad {}_sW^{(6)}_{\alpha_1},\ {}_s\dot{W}^{(6)}_{\alpha_1} \doteqdot (+1)\times\ldots$$

with $$(H+z) \quad \text{for} \quad (H-z),$$

$${}_s\phi^{(7)}_{\alpha_1} \doteqdot (+1)\times\ldots,\quad {}_sU^{(7)}_{\alpha_1},\ {}_s\dot{U}^{(7)}_{\alpha_1} \doteqdot (+1)\times\ldots,\quad {}_sW^{(7)}_{\alpha_1},\ {}_s\dot{W}^{(7)}_{\alpha_1} \doteqdot (-1)\times\ldots$$

with $$(H+h) \text{ for } (H-h) \quad \text{and} \quad (H+z) \text{ for } (H-z).$$

${}_s\psi$ contributions from Γ_{α_1}

$$\begin{cases} {}_s\psi^{(1)}_{\alpha_1} \doteqdot \dfrac{-8\surd(2\pi i)\,[x\alpha_1 - (h+z)\,\widehat{\beta_1\alpha_1}]^{-\frac{3}{2}}}{\alpha_1(2/\alpha_1^2 - 1/\beta_1^2)^2}\,\omega^{-\frac{3}{2}}\exp\{i\omega\tau\}, \\ {}_sU^{(1)}_{\alpha_1} \doteqdot \dfrac{-16\surd 2\,[x\alpha_1 - (h+z)\,\widehat{\beta_1\alpha_1}]^{-\frac{3}{2}}}{\alpha_1\widehat{\beta_1\alpha_1}(2/\alpha_1^2 - 1/\beta_1^2)^2}\,\tau^{\frac{1}{2}}H(\tau) \doteqdot -{}_sW^{(1)}_{\alpha_1}\times\alpha_1/\widehat{\beta_1\alpha_1}, \end{cases}$$

where $$\tau = t - x/\alpha_1 - (h+z)/\widehat{\beta_1\alpha_1},$$

and similarly $${}_s\psi^{(2)}_{\alpha_1},\quad {}_sU^{(2)}_{\alpha_1},\quad {}_sW^{(2)}_{\alpha_1} = (+2)\times\ldots,$$

where $$\tau = t - x/\alpha_1 - (h+z)/\widehat{\beta_1\alpha_1} - 2H^2/[x\alpha_1 - (h+z)\,\widehat{\beta_1\alpha_1}];$$

$$\left\{\begin{aligned}
&{}_s\psi^{(3)}_{\alpha_1} \doteqdot \frac{4\surd(2\pi\mathrm{i})\,\alpha_1[x\alpha_1-(H+h-z)\,\widehat{\beta_1\alpha_1}]^{-\frac{3}{2}}}{(2/\alpha_1^2-1/\beta_1^2)}\left[\frac{Y_1}{S_0}\right]_{\alpha_1}\omega^{-\frac{3}{2}}\exp\{\mathrm{i}\omega\tau\},\\
&\left\{\begin{aligned}
&{}_sU^{(3)}_{\alpha_1} \doteqdot \frac{-8\surd 2\,\alpha_1[x\alpha_1-(H+h-z)\,\widehat{\beta_1\alpha_1}]^{-\frac{3}{2}}}{\widehat{\beta_1\alpha_1}(2/\alpha_1^2-1/\beta_1^2)}\left[\frac{Y_1}{S_0}\right]_{\alpha_1,0}\tau^{\frac{1}{2}}H(\tau),\\
&{}_s\dot{U}^{(3)}_{\alpha_1} \doteqdot \frac{-4\surd 2\,\mathrm{i}\,\alpha_1[x\alpha_1-(H+h-z)\,\widehat{\beta_1\alpha_1}]^{-\frac{3}{2}}}{\widehat{\beta_1\alpha_1}(2/\alpha_1^2-1/\beta_1^2)}\left[\frac{Y_1}{S_0}\right]_{\alpha_1,1}\tau'^{-\frac{1}{2}}H(\tau'),
\end{aligned}\right.\\
&\left\{\begin{aligned}
&{}_sW^{(3)}_{\alpha_1} \doteqdot \frac{-8\surd 2\,[x\alpha_1-(H+h-z)\,\widehat{\beta_1\alpha_1}]^{-\frac{3}{2}}}{(2/\alpha_1^2-1/\beta_1^2)}\left[\frac{Y_1}{S_0}\right]_{\alpha_1,0}\tau^{\frac{1}{2}}H(\tau),\\
&{}_s\dot{W}^{(3)}_{\alpha_1} \doteqdot \frac{4\surd 2\,\mathrm{i}\,[x\alpha_1-(H+h-z)\,\widehat{\beta_1\alpha_1}]^{-\frac{3}{2}}}{(2/\alpha_1^2-1/\beta_1^2)}\left[\frac{Y_1}{S_0}\right]_{\alpha_1,1}\tau'^{-\frac{1}{2}}H(\tau'),
\end{aligned}\right.
\end{aligned}\right.$$

where $\tau = -\tau' = t-x/\alpha_1-(H+h-z)/\widehat{\beta_1\alpha_1}-H^2/2[x\alpha_1-(H+h-z)\,\widehat{\beta_1\alpha_1}],$

and similarly

$${}_s\psi^{(4)}_{\alpha_1} \doteqdot (+1)\times\ldots,\quad {}_sU^{(4)}_{\alpha_1},\ {}_s\dot{U}^{(4)}_{\alpha_1} \doteqdot (-1)\times\ldots,\quad {}_sW^{(4)}_{\alpha_1},\ {}_s\dot{W}^{(4)}_{\alpha_1} \doteqdot (+1)\times\ldots$$

with $(H-h+z)$ for $(H+h-z)$,

$${}_s\psi^{(5)}_{\alpha_1} \doteqdot (-2)\times\ldots,\quad {}_sU^{(5)}_{\alpha_1},\ {}_s\dot{U}^{(5)}_{\alpha_1} \doteqdot (+2)\times\ldots,\quad {}_sW^{(5)}_{\alpha_1},\ {}_s\dot{W}^{(5)}_{\alpha_1} \doteqdot (-2)\times\ldots$$

with $(H+h+z)$ for $(H+h-z)$;

$$\left\{\begin{aligned}
&{}_s\psi^{(6)}_{\alpha_1} \doteqdot \surd(2\pi\mathrm{i}^3)\,\widehat{\beta_1\alpha_1}\,\alpha_1^2[x\alpha_1-(2H-h-z)\,\widehat{\beta_1\alpha_1}]^{-\frac{3}{2}}\,[T_1/S_0-T_0S_1/S_0^2]_{\alpha_1}\,\omega^{-\frac{3}{2}}\exp\{\mathrm{i}\omega\tau\},\\
&\left\{\begin{aligned}
&{}_sU^{(6)}_{\alpha_1} \doteqdot -2\surd 2\,\mathrm{i}\,\alpha_1^2[x\alpha_1-(2H-h-z)\,\widehat{\beta_1\alpha_1}]^{-\frac{3}{2}}\,[T_1/S_0-T_0S_1/S^2]_{\alpha_1,0}\,\tau^{\frac{1}{2}}H(\tau),\\
&{}_s\dot{U}^{(6)}_{\alpha_1} \doteqdot \surd 2\,\alpha_1^2[x\alpha_1-(2H-h-z)\,\widehat{\beta_1\alpha_1}]^{-\frac{3}{2}}\,[T_1/S_0-T_0S_1/S_0^2]_{\alpha_1,1}\,\tau'^{-\frac{1}{2}}H(\tau'),
\end{aligned}\right.\\
&\left\{\begin{aligned}
&{}_sW^{(6)}_{\alpha_1} \doteqdot -2\surd 2\,\mathrm{i}\,\widehat{\beta_1\alpha_1}\,\alpha_1[x\alpha_1-(2H-h-z)\,\widehat{\beta_1\alpha_1}]^{-\frac{3}{2}}\,[T_1/S_0-T_0S_1/S_0^2]_{\alpha_1,0}\,\tau^{\frac{1}{2}}H(\tau),\\
&{}_s\dot{W}^{(6)}_{\alpha_1} \doteqdot -\surd 2\,\widehat{\beta_1\alpha_1}\,\alpha_1[x\alpha_1-(2H-h-z)\,\widehat{\beta_1\alpha_1}]^{-\frac{3}{2}}\,[T_1/S_0-T_0S_1/S_0^2]_{\alpha_1,1}\,\tau'^{-\frac{1}{2}}H(\tau'),
\end{aligned}\right.
\end{aligned}\right.$$

where $\tau = -\tau' = t-x/\alpha_1-(2H-h-z)/\widehat{\beta_1\alpha_1},$

and similarly

$${}_s\psi^{(7)}_{\alpha_1} \doteqdot (-1)\times\ldots,\quad {}_sU^{(7)}_{\alpha_1},\ {}_s\dot{U}^{(7)}_{\alpha_1} \doteqdot (-1)\times\ldots,\quad {}_sW^{(7)}_{\alpha_1},\ {}_s\dot{W}^{(7)}_{\alpha_1} \doteqdot (-1)\times\ldots$$

with $(2H+h-z)$ for $(2H-h-z)$

and $[T_1/S_0-T_0S_1/S_0^2-2F_1T_0/F_0S_0]$ for $[T_1/S_0-T_0S_1/S_0^2]$,

$${}_s\psi^{(8)}_{\alpha_1} \doteqdot (-1)\times\ldots,\quad {}_sU^{(8)}_{\alpha_1},\ {}_s\dot{U}^{(8)}_{\alpha_1} \doteqdot (+1)\times\ldots,\quad {}_sW^{(8)}_{\alpha_1},\ {}_s\dot{W}^{(8)}_{\alpha_1} \doteqdot (-1)\times\ldots$$

with $(2H-h+z)$ for $(2H-h-z)$

and $[T_1/S_0-T_0S_1/S_0^2-2F_1T_0/F_0S_0]$ for $[T_1/S_0-T_0S_1/S_0^2]$,

$${}_s\psi^{(9)}_{\alpha_1} \doteqdot (+1)\times\ldots,\quad {}_sU^{(9)}_{\alpha_1},\ {}_s\dot{U}^{(9)}_{\alpha_1} \doteqdot (-1)\times\ldots,\quad {}_sW^{(9)}_{\alpha_1},\ {}_s\dot{W}^{(9)}_{\alpha_1} \doteqdot (+1)\times\ldots$$

with $(2H+h+z)$ for $(2H-h-z)$

and $[T_1/S_0-T_0S_1/S_0^2-4F_1T_0/F_0S_0]$ for $[T_1/S_0-T_0S_1/S_0^2]$.

${}_s\phi$ contributions from Γ_{β_2}

$$\begin{cases} {}_s\phi^{(2)}_{\beta_2} \doteqdot \dfrac{4\sqrt{(2\pi i^3)}\,\beta_2^5(2/\beta_2^2-1/\beta_1^2)}{[x\beta_2-(2H-h)\,\widehat{\beta_1\beta_2}]^{\frac{3}{2}}}\left[\dfrac{1}{F_0}\left(\dfrac{T_1}{S_0}-\dfrac{T_0S_1}{S_0^2}\right)\right]_{\beta_2}\exp\{i\omega\tau-\omega p\}, \\ {}_s\dot{U}^{(2)}_{\beta_2} \doteqdot \dfrac{4\sqrt{2}\,i\,\beta_2^4(2/\beta_2^2-1/\beta_1^2)}{[x\beta_2-(2H-h)\,\widehat{\beta_1\beta_2}]^{\frac{3}{2}}}\left(\dfrac{z}{\widehat{\beta_2\alpha_1}}\right)^{-\frac{1}{2}}\begin{Bmatrix} i \\ 1\end{Bmatrix}\left[\dfrac{1}{F_0}\left(\dfrac{T_1}{S_0}-\dfrac{T_0S_1}{S_0^2}\right)\right]_{\beta_2,\,{1, \atop 0,}}\sin(\tfrac{1}{2}\psi\mp\tfrac{1}{4}\pi)\cos^{\frac{1}{2}}\psi, \\ {}_s\dot{W}^{(2)}_{\beta_2} \doteqdot \dfrac{-4\sqrt{2}\,i\,\beta_2^5(2/\beta_2^2-1/\beta_1^2)}{\widehat{\beta_2\alpha_1}[x\beta_2-(2H-h)\,\widehat{\beta_1\beta_2}]^{\frac{3}{2}}}\left(\dfrac{z}{\widehat{\beta_2\alpha_1}}\right)^{-\frac{1}{2}}\begin{Bmatrix} 1 \\ -i\end{Bmatrix}\left[\dfrac{1}{F_0}\left(\dfrac{T_1}{S_0}-\dfrac{T_0S_1}{S_0^2}\right)\right]_{\beta_2,\,{0, \atop 1,}}\sin(\tfrac{1}{2}\psi\mp\tfrac{1}{4}\pi)\cos^{\frac{1}{2}}\psi, \end{cases}$$

where $\tau = t-x/\beta_2-(2H-h)/\widehat{\beta_1\beta_2}, \quad p = z/\widehat{\beta_2\alpha_1}, \quad \tan\psi = \tau/p,$

and similarly

$${}_s\phi^{(3)}_{\beta_2} \doteqdot (-1)\times\ldots, \quad {}_sU^{(3)}_{\beta_2} \doteqdot (-1)\times\ldots, \quad {}_sW^{(3)}_{\beta_2} \doteqdot (-1)\times\ldots$$

with $(2H+h)$ for $(2H-h)$ and $(\overline{F}_0/F_0^2]$ for $[1/F_0]$;

$$\begin{cases} {}_s\phi^{(4)}_{\beta_2} \doteqdot \dfrac{-\sqrt{(2\pi i^3)}\,\widehat{\beta_2\alpha_1}\,\beta_2^2}{[x\beta_2-(H-h)\,\widehat{\beta_1\beta_2}]^{\frac{3}{2}}}\left[\dfrac{Y_1}{S_0}-\dfrac{Y_0S_1}{S_0^2}\right]_{\beta_2}\exp\{i\omega\tau-\omega p\}, \\ {}_s\dot{U}^{(4)}_{\beta_2} \doteqdot \dfrac{-\sqrt{2}\,i\,\widehat{\beta_2\alpha_1}\,\beta_2}{[x\beta_2-(H-h)\,\widehat{\beta_1\beta_2}]^{\frac{3}{2}}}\left(\dfrac{H-z}{\widehat{\beta_2\alpha_1}}\right)^{-\frac{1}{2}}\begin{Bmatrix} i \\ 1\end{Bmatrix}\left[\dfrac{Y_1}{S_0}-\dfrac{Y_0S_1}{S_0^2}\right]_{\beta_2,\,{0, \atop 1,}}\sin(\tfrac{1}{2}\psi\mp\tfrac{1}{4}\pi)\cos^{\frac{1}{2}}\psi, \\ {}_s\dot{W}^{(4)}_{\beta_2} \doteqdot \dfrac{-\sqrt{2}\,i\,\beta_2^2}{[x\beta_2-(H-h)\,\widehat{\beta_1\beta_2}]^{\frac{3}{2}}}\left(\dfrac{H-z}{\widehat{\beta_2\alpha_1}}\right)^{-\frac{1}{2}}\begin{Bmatrix} 1 \\ -i\end{Bmatrix}\left[\dfrac{Y_1}{S_0}-\dfrac{Y_0S_1}{S_0^2}\right]_{\beta_2,\,{1, \atop 0,}}\sin(\tfrac{1}{2}\psi\mp\tfrac{1}{4}\pi)\cos^{\frac{1}{2}}\psi, \end{cases}$$

where $\tau = t-x/\beta_2-(H-h)/\widehat{\beta_1\beta_2}, \quad p = (H-z)/\widehat{\beta_2\alpha_1}, \quad \tan\psi = \tau/p,$

and similarly

$${}_s\phi^{(5)}_{\beta_2} \doteqdot (-1)\times\ldots, \quad {}_s\dot{U}^{(5)}_{\beta_2} \doteqdot (-1)\times\ldots, \quad {}_s\dot{W}^{(5)}_{\beta_2} \doteqdot (-1)\times\ldots$$

with $(H+h)$ for $(H-h)$ and $[(Y_1/S_0-Y_0S_1/S_0^2)\,\overline{F}_0/F_0]$ for $[Y_1/S_0-Y_0S_1/S_0^2]$,

$${}_s\phi^{(6)}_{\beta_2} \doteqdot (-1)\times\ldots, \quad {}_s\dot{U}^{(6)}_{\beta_2} \doteqdot (-1)\times\ldots, \quad {}_s\dot{W}^{(6)}_{\beta_2} \doteqdot (+1)\times\ldots$$

with $(H+z)$ for $(H-z)$ and $[(Y_1/S_0-Y_0S_1/S_0^2)\,\overline{F}_0/F_0]$ for $[Y_1/S_0-Y_0S_1/S_0^2]$,

$${}_s\phi^{(7)}_{\beta_2} \doteqdot (+1)\times\ldots, \quad {}_s\dot{U}^{(7)}_{\beta_2} \doteqdot (+1)\times\ldots, \quad {}_s\dot{W}^{(7)}_{\beta_2} \doteqdot (-1)\times\ldots$$

with $(H+h)$ for $(H-h)$, $(H+z)$ for $(H-z)$

and $[(Y_1/S_0-Y_0S_1/S_0^2)\,(\overline{F}_0^2/F_0^2+16i\beta_2^2/\widehat{\beta_1\beta_2}\,\widehat{\beta_2\alpha_1}\,F_0^2)]$ for $[Y_1/S_0-Y_0S_1/S_0^2]$;

$$\begin{cases} {}_s\phi^{(8)}_{\beta_2} \doteqdot \dfrac{4\sqrt{(2\pi i^3)}\,\beta_2^5(2/\beta_2^2-1/\beta_1^2)}{[x\beta_2-h\widehat{\beta_1\beta_2}]^{\frac{3}{2}}}\left[\dfrac{1}{F_0}\left(\dfrac{W_1}{S_0}-\dfrac{W_0S_1}{S_0^2}\right)\right]_{\beta_2}\exp\{i\omega\tau-\omega p\}, \\ {}_s\dot{U}^{(8)}_{\beta_2} \doteqdot \dfrac{4\sqrt{2}\,i\beta_2^4(2/\beta_2^2-1/\beta_1^2)}{[x\beta_2-h\widehat{\beta_1\beta_2}]^{\frac{3}{2}}}\left(\dfrac{2H-z}{\widehat{\beta_2\alpha_1}}\right)^{-\frac{1}{2}}\begin{Bmatrix} i \\ 1\end{Bmatrix}\left[\dfrac{1}{F_0}\left(\dfrac{W_1}{S_0}-\dfrac{W_0S_1}{S_0^2}\right)\right]_{\beta_2,\,{1, \atop 0,}}\sin(\tfrac{1}{2}\psi\mp\tfrac{1}{4}\pi)\cos^{\frac{1}{2}}\psi, \\ {}_s\dot{W}^{(8)}_{\beta_2} \doteqdot \dfrac{4\sqrt{2}\,i\beta_2^5(2/\beta_2^2-1/\beta_1^2)}{[x\beta_2-h\widehat{\beta_1\beta_2}]^{\frac{3}{2}}}\left(\dfrac{2H-z}{\widehat{\beta_2\alpha_1}}\right)^{-\frac{1}{2}}\begin{Bmatrix} 1 \\ -i\end{Bmatrix}\left[\dfrac{1}{F_0}\left(\dfrac{W_1}{S_0}-\dfrac{W_0S_1}{S_0^2}\right)\right]_{\beta_2,\,{0, \atop 1,}}\sin(\tfrac{1}{2}\psi\mp\tfrac{1}{4}\pi)\cos^{\frac{1}{2}}\psi, \end{cases}$$

where $\tau = t-x/\beta_2-h/\widehat{\beta_1\beta_2}, \quad p = (2H-z)/\widehat{\beta_2\alpha_1}, \quad \tan\psi = \tau/p,$

and similarly

$${}_s\phi^{(9)}_{\beta_2} \doteqdot (-1)\times\ldots,\quad {}_s\dot{U}^{(9)}_{\beta_2} \doteqdot (-1)\times\ldots,\quad {}_s\dot{W}^{(9)}_{\beta_2} \doteqdot (+1)\times\ldots$$

with $(2H+z)$ for $(2H-z)$ and $[\overline{F}_0/F_0^2]$ for $[1/F_0]$.

${}_s\psi$ contributions from Γ_{β_2}

$$\begin{cases} {}_s\psi^{(2)}_{\beta_2} \doteqdot \dfrac{-16\sqrt{(2\pi i)}\,\beta_2^8(2/\beta_2^2-1/\beta_1^2)^2}{\widehat{\beta_2\alpha_1}[x\beta_2-(h+z)\,\widehat{\beta_1\beta_2}]^{\frac{3}{2}}}\left[\dfrac{1}{F_0^2}\left(\dfrac{W_1}{S_0}-\dfrac{W_0S_1}{S_0^2}\right)\right]_{\beta_2}\exp\{i\omega\tau-\omega p\}, \\ {}_s\dot{U}^{(2)}_{\beta_2} \doteqdot \dfrac{16\sqrt{2}\,\beta_2^8(2/\beta_2^2-1/\beta_1^2)^2}{\widehat{\beta_2\alpha_1}\,\widehat{\beta_1\beta_2}[x\beta_2-(h+z)\,\widehat{\beta_1\beta_2}]^{\frac{3}{2}}}\left(\dfrac{2H}{\widehat{\beta_2\alpha_1}}\right)^{-\frac{1}{2}}\begin{Bmatrix} i\\ 1\end{Bmatrix}\left[\dfrac{1}{F_0^2}\left(\dfrac{W_1}{S_0}-\dfrac{W_0S_1}{S_0^2}\right)\right]_{\beta_2,\,{0,\atop 1,}}\sin(\tfrac{1}{2}\psi\mp\tfrac{1}{4}\pi)\cos^{\frac{1}{2}}\psi, \\ {}_s\dot{W}^{(2)}_{\beta_2} \doteqdot \dfrac{16\sqrt{2}\,\beta_2^7(2/\beta_2^2-1/\beta_1^2)}{\widehat{\beta_2\alpha_1}[x\beta_2-(h+z)\,\widehat{\beta_1\beta_2}]^{\frac{3}{2}}}\left(\dfrac{2H}{\widehat{\beta_2\alpha_1}}\right)^{-\frac{1}{2}}\begin{Bmatrix} i\\ 1\end{Bmatrix}\left[\dfrac{1}{F_0^2}\left(\dfrac{W_1}{S_0}-\dfrac{W_0S_1}{S_0^2}\right)\right]_{\beta_2,\,{1,\atop 0,}}\sin(\tfrac{1}{2}\psi\mp\tfrac{1}{4}\pi)\cos^{\frac{1}{2}}\psi, \end{cases}$$

where $\tau = t-x/\beta_2-(h+z)/\widehat{\beta_1\beta_2},\quad p = 2H/\widehat{\beta_2\alpha_1},\quad \tan\psi = \tau/p;$

$$\begin{cases} {}_s\psi^{(3)}_{\beta_2} \doteqdot \dfrac{4\sqrt{(2\pi i)}\,\beta_2^5(2/\beta_2^2-1/\beta_1^2)}{[x\beta_2-(H+h-z)\,\widehat{\beta_1\beta_2}]^{\frac{3}{2}}}\left[\dfrac{1}{F_0}\left(\dfrac{Y_1}{S_0}-\dfrac{Y_0S_1}{S_0^2}\right)\right]_{\beta_2}\exp\{i\omega\tau-\omega p\}, \\ {}_s\dot{U}^{(3)}_{\beta_2} \doteqdot \dfrac{4\sqrt{2}\,\beta_2^5(2/\beta_2^2-1/\beta_1^2)}{\widehat{\beta_1\beta_2}[x\beta_2-(H+h-z)\,\widehat{\beta_1\beta_2}]^{\frac{3}{2}}}\left(\dfrac{H}{\widehat{\beta_2\alpha_1}}\right)^{-\frac{1}{2}}\begin{Bmatrix} i\\ 1\end{Bmatrix}\left[\dfrac{1}{F_0}\left(\dfrac{Y_1}{S_0}-\dfrac{Y_0S_1}{S_0^2}\right)\right]_{\beta_2,\,{1,\atop 0,}}\sin(\tfrac{1}{2}\psi\mp\tfrac{1}{4}\pi)\cos^{\frac{1}{2}}\psi, \\ {}_s\dot{W}^{(3)}_{\beta_2} \doteqdot \dfrac{-4\sqrt{2}\,\beta_2^4(2/\beta_2^2-1/\beta_1^2)}{[x\beta_2-(H+h-z)\,\widehat{\beta_1\beta_2}]^{\frac{3}{2}}}\left(\dfrac{H}{\widehat{\beta_2\alpha_1}}\right)^{-\frac{1}{2}}\begin{Bmatrix} 1\\ -i\end{Bmatrix}\left[\dfrac{1}{F_0}\left(\dfrac{Y_1}{S_0}-\dfrac{Y_0S_1}{S_0^2}\right)\right]_{\beta_2,\,{0,\atop 1,}}\sin(\tfrac{1}{2}\psi\mp\tfrac{1}{4}\pi)\cos^{\frac{1}{2}}\psi, \end{cases}$$

where $\tau = t-x/\beta_2-(H+h-z)/\widehat{\beta_1\beta_2},\quad p = H/\widehat{\beta_2\alpha_1},\quad \tan\psi = \tau/p,$

and similarly

$${}_s\psi^{(4)}_{\beta_2} \doteqdot (+1)\times\ldots,\quad {}_s\dot{U}^{(4)}_{\beta_2} \doteqdot (-1)\times\ldots,\quad {}_s\dot{W}^{(4)}_{\beta_2} \doteqdot (+1)\times\ldots$$

with $(H-h+z)$ for $(H+h-z)$,

$${}_s\psi^{(5)}_{\beta_2} \doteqdot (-2)\times\ldots,\quad {}_s\dot{U}^{(5)}_{\beta_2} \doteqdot (+2)\times\ldots,\quad {}_s\dot{W}^{(5)}_{\beta_2} \doteqdot (-2)\times\ldots$$

with $(H+h+z)$ for $(H+h-z)$ and $[\overline{F}_0/F_0^2]$ for $[1/F_0]$;

$$\begin{cases} {}_s\psi^{(6)}_{\beta_2} \doteqdot \sqrt{(2\pi i^3)}\,\beta_2^2\widehat{\beta_1\beta_2}[x\beta_2-(2H-h-z)\,\widehat{\beta_1\beta_2}]^{-\frac{3}{2}}[T_1/S_0-T_0S_1/S_0^2]_{\beta_2}\exp\{i\omega\tau-\omega p\}, \\ \begin{cases} {}_sU^{(6)}_{\beta_2} \doteqdot -2\sqrt{2}\,i\,\beta_2^2[x\beta_2-(2H-h-z)\,\widehat{\beta_1\beta_2}]^{-\frac{3}{2}}[T_1/S_0-T_0S_1/S_0^2]_{\beta_2,0}\,\tau^{\frac{1}{2}}H(\tau), \\ {}_s\dot{U}^{(6)}_{\beta_2} \doteqdot -\sqrt{2}\,\beta_2^2[x\beta_2-(2H-h-z)\,\widehat{\beta_1\beta_2}]^{-\frac{3}{2}}[T_1/S_0-T_0S_1/S_0^2]_{\beta_2,1}\,\tau'^{-\frac{1}{2}}H(\tau'), \end{cases} \\ \begin{cases} {}_sW^{(6)}_{\beta_2} \doteqdot -2\sqrt{2}\,i\,\beta_2\widehat{\beta_1\beta_2}[x\beta_2-(2H-h-z)\,\widehat{\beta_1\beta_2}]^{-\frac{3}{2}}[T_1/S_0-T_0S_1/S_0^2]_{\beta_2,0}\,\tau^{\frac{1}{2}}H(\tau), \\ {}_s\dot{W}^{(6)}_{\beta_2} \doteqdot -\sqrt{2}\,\beta_2\widehat{\beta_1\beta_2}[x\beta_2-(2H-h-z)\,\widehat{\beta_1\beta_2}]^{-\frac{3}{2}}[T_1/S_0-T_0S_1/S_0^2]_{\beta_2,1}\,\tau'^{-\frac{1}{2}}H(\tau'), \end{cases} \end{cases}$$

where $\tau = -\tau' = t-x/\beta_2-(2H-h-z)/\widehat{\beta_1\beta_2},$

and similarly

$$_s\psi^{(7)}_{\beta_2} \doteqdot (-1)\times\ldots,\quad {}_sU^{(7)}_{\beta_2},\ {}_s\dot{U}^{(7)}_{\beta_2} \doteqdot (-1)\times\ldots,\quad {}_sW^{(7)}_{\beta_2},\ {}_s\dot{W}^{(7)}_{\beta_2} \doteqdot (-1)\times\ldots$$

with $(2H+h-z)$ for $(2H-h-z)$

and $[(T_1/S_0-T_0S_1/S_0^2)\,\overline{F}_0/F_0]$ for $[T_1/S_0-T_0S_1/S_0^2]$,

$$_s\psi^{(8)}_{\beta_2} \doteqdot (-1)\times\ldots,\quad {}_sU^{(8)}_{\beta_2},\ {}_s\dot{U}^{(8)}_{\beta_2} \doteqdot (+1)\times\ldots,\quad {}_sW^{(8)}_{\beta_2},\ {}_s\dot{W}^{(8)}_{\beta_2} \doteqdot (-1)\times\ldots$$

with $(2H-h+z)$ for $(2H-h-z)$

and $[(T_1/S_0-T_0S_1/S_0^2)\,\overline{F}_0/F_0]$ for $[T_1/S_0-T_0S_1/S_0^2]$,

$$_s\psi^{(9)}_{\beta_2} \doteqdot (+1)\times\ldots,\quad {}_sU^{(9)}_{\beta_2},\ {}_s\dot{U}^{(9)}_{\beta_2} \doteqdot (-1)\times\ldots,\quad {}_sW^{(9)}_{\beta_2},\ {}_s\dot{W}^{(9)}_{\beta_2} \doteqdot (+1)\times\ldots$$

with $(2H+h+z)$ for $(2H-h-z)$

and $[(T_1/S_0-T_0S_1/S_0^2)\,\overline{F}{}_0^2/F_0^2]$ for $[T_1/S_0-T_0S_1/S_0^2]$.

${}_s\phi$ contributions from Γ_{β_1}

$$\left\{\begin{aligned}
&{}_s\phi^{(1)}_{\beta_1} \doteqdot 4\sqrt{(2\pi \mathrm{i}^3)}\,\beta_1^{\frac{1}{2}}x^{-\frac{1}{2}}\{h\omega^{-\frac{1}{2}}+\beta_1[F_1/F_0]_{\beta_1}\omega^{-\frac{3}{2}}\}\exp\{\mathrm{i}\omega\tau-\omega p\},\\
&\left\{\begin{aligned}&{}_sU^{(1)}_{\beta_1} \doteqdot 4\sqrt{2}\,\beta_1^{-\frac{1}{2}}x^{-\frac{1}{2}}(z/\widehat{\beta_1\alpha_1})^{-\frac{1}{2}}h\sin(\tfrac{1}{2}\psi+\tfrac{1}{4}\pi)\cos^{\frac{1}{2}}\psi,\\
&{}_s\dot{U}^{(1)}_{\beta_1} \doteqdot -4\sqrt{2}\,\beta_1^{\frac{1}{2}}x^{-\frac{1}{2}}(z/\widehat{\beta_1\alpha_1})^{-\frac{1}{2}}[F_1/F_0]_{\beta_1,1}\sin(\tfrac{1}{2}\psi-\tfrac{1}{4}\pi)\cos^{\frac{1}{2}}\psi,\end{aligned}\right.\\
&\left\{\begin{aligned}&{}_sW^{(1)}_{\beta_1} \doteqdot 4\sqrt{2}\,(\beta_1^{\frac{1}{2}}/\widehat{\beta_1\alpha_1})\,x^{-\frac{1}{2}}(z/\widehat{\beta_1\alpha_1})^{-\frac{1}{2}}h\sin(\tfrac{1}{2}\psi-\tfrac{1}{4}\pi)\cos^{\frac{1}{2}}\psi,\\
&{}_s\dot{W}^{(1)}_{\beta_1} \doteqdot 4\sqrt{2}\,(\beta_1^{\frac{3}{2}}/\widehat{\beta_1\alpha_1})\,x^{-\frac{1}{2}}(z/\widehat{\beta_1\alpha_1})^{-\frac{1}{2}}[F_1/F_0]_{\beta_1,1}\sin(\tfrac{1}{2}\psi+\tfrac{1}{4}\pi)\cos^{\frac{1}{2}}\psi,\end{aligned}\right.
\end{aligned}\right.$$

where $\tau = t-x/\beta_1-h^2/2x\beta_1,\quad p = z/\widehat{\beta_1\alpha_1},\quad \tan\psi=\tau/p,$

and similarly

$$_s\phi^{(2)}_{\beta_1} \doteqdot (-1)\times\ldots,\quad {}_sU^{(2)}_{\beta_1},\ {}_s\dot{U}^{(2)}_{\beta_1} \doteqdot (-1)\times\ldots,\quad {}_sW^{(2)}_{\beta_1},\ {}_s\dot{W}^{(2)}_{\beta} \doteqdot (-1)\times\ldots$$

with $(2H-h)$ for h and $[2S_1/S_0+F_1/F_0]$ for $[F_1/F_0]$,

$$_s\phi^{(3)}_{\beta_1} \doteqdot (+1)\times\ldots,\quad {}_sU^{(3)}_{\beta_1},\ {}_s\dot{U}^{(3)}_{\beta_1} \doteqdot (+1)\times\ldots,\quad {}_sW^{(3)}_{\beta_1},\ {}_sW^{(3)}_{\beta_1} \doteqdot (+1)\times\ldots$$

with $(2H+h)$ for h and $[2S_1/S_0+3F_1/F_0]$ for $[F_1/F_0]$;

$$\left\{\begin{aligned}
&{}_s\phi^{(4)}_{\beta_1} \doteqdot \sqrt{(2\pi \mathrm{i}^3)}\,\beta_1^{-\frac{1}{2}}\widehat{\beta_1\alpha_1}\,x^{-\frac{1}{2}}\{(H-h)\,[Y_0/S_0]_{\beta_1}\omega^{-\frac{1}{2}}-\beta_1[Y_1/S_0-Y_0S_1/S_0^2]\,\omega^{-\frac{3}{2}}\}\exp\{\mathrm{i}\omega\tau-\omega p\},\\
&\left\{\begin{aligned}&{}_sU^{(4)}_{\beta_1} \doteqdot \frac{\sqrt{2}\,\widehat{\beta_1\alpha_1}(H-h)}{\beta_1^{\frac{1}{2}}x^{\frac{1}{2}}}\left(\frac{H-z}{\widehat{\beta_1\alpha_1}}\right)^{-\frac{1}{2}}\left[\frac{Y_0}{S_0}\right]_{\beta_1,1}\sin(\tfrac{1}{2}\psi+\tfrac{1}{4}\pi)\cos^{\frac{1}{2}}\psi,\\
&{}_s\dot{U}^{(4)}_{\beta_1} \doteqdot \frac{-\sqrt{2}\,\widehat{\beta_1\alpha_1}}{\beta_1^{\frac{1}{2}}x^{\frac{1}{2}}}\left(\frac{H-z}{\widehat{\beta_1\alpha_1}}\right)^{-\frac{1}{2}}\left[\frac{Y_0S_1}{S_0^2}\right]_{\beta_1,0}\sin(\tfrac{1}{2}\psi-\tfrac{1}{4}\pi)\cos^{\frac{1}{2}}\psi,\end{aligned}\right.\\
&\left\{\begin{aligned}&{}_sW^{(4)}_{\beta_1} \doteqdot \sqrt{2}\,\beta_1^{-\frac{1}{2}}x^{-\frac{1}{2}}(H-h)\left(\frac{H-z}{\widehat{\beta_1\alpha_1}}\right)^{-\frac{1}{2}}\left[\frac{Y_0}{S_0}\right]_{\beta_1,1}\sin(\tfrac{1}{2}\psi-\tfrac{1}{4}\pi)\cos^{\frac{1}{2}}\psi,\\
&{}_s\dot{W}^{(4)}_{\beta_1} \doteqdot -\sqrt{2}\,\beta_1^{\frac{1}{2}}x^{-\frac{1}{2}}\left(\frac{H-z}{\widehat{\beta_1\alpha_1}}\right)^{-\frac{1}{2}}\left[\frac{Y_0S_1}{S_0^2}\right]_{\beta_1,0}\sin(\tfrac{1}{2}\psi+\tfrac{1}{4}\pi)\cos^{\frac{1}{2}}\psi,\end{aligned}\right.
\end{aligned}\right.$$

where $\tau = t-x/\beta_1-(H-h)^2/2x\beta_1,\quad p=(H-z)/\widehat{\beta_1\alpha_1},\quad \tan\psi=\tau/p,$

and similarly

$$ {}_s\phi^{(5)}_{\beta_1} \doteqdot (-1)\times\ldots,\quad {}_sU^{(5)}_{\beta_1},\ {}_s\dot{U}^{(5)}_{\beta_1} \doteqdot (-1)\times\ldots,\quad {}_sW^{(5)}_{\beta_1},\ {}_s\dot{W}^{(5)}_{\beta_1} \doteqdot (-1)\times\ldots $$

with $(H+h)$ for $(H-h)$ and $[Y_0S_1/S_0^2+2F_1Y_0/F_0S_0]$ for $[Y_0S_1/S_0^2]$,

$$ {}_s\phi^{(6)}_{\beta_1} \doteqdot (-1)\times\ldots,\quad {}_sU^{(6)}_{\beta_1},\ {}_s\dot{U}^{(6)}_{\beta_1} \doteqdot (-1)\times\ldots,\quad {}_sW^{(6)}_{\beta_1},\ {}_s\dot{W}^{(6)}_{\beta_1} \doteqdot (+1)\times\ldots $$

with $(H+z)$ for $(H-z)$ and $[Y_0S_1/S_0^2+2F_1Y_0/F_0S_0]$ for $[Y_0S_1/S_0^2]$,

$$ {}_s\phi^{(7)}_{\beta_1} \doteqdot (+1)\times\ldots,\quad {}_sU^{(7)}_{\beta_1},\ {}_s\dot{U}^{(7)}_{\beta_1} \doteqdot (+1)\times\ldots,\quad {}_sW^{(7)}_{\beta_1},\ {}_s\dot{W}^{(7)}_{\beta_1} \doteqdot (-1)\times\ldots $$

with $(H+h)$ for $(H-h)$, $(H+z)$ for $(H-z)$

and $[Y_0S_1/S_0^2+4F_1Y_0/F_0S_0+16\beta_1Y_0/\widehat{\beta_1\alpha_1}S_0]$ for $[Y_0S_1/S_0^2]$;

$$ \left\{ \begin{aligned} {}_s\phi^{(8)}_{\beta_1} &\doteqdot -4\sqrt{(2\pi i^3)}\,\beta_1^{\frac{1}{2}}x^{-\frac{3}{2}}\{h\omega^{-\frac{1}{2}}[W_0/S_0]_{\beta_1}-\beta_1\omega^{-\frac{3}{2}}[(W_1/W_0-S_1/S_0-F_1/F_0)\,W_0/S_0]_{\beta_1}\} \\ &\qquad\times\exp\{i\omega\tau-\omega p\}, \\ {}_sU^{(8)}_{\beta_1} &\doteqdot \frac{-4\sqrt{2}\,h}{\beta_1^{\frac{1}{2}}x^{\frac{3}{2}}}\left(\frac{2H-z}{\widehat{\beta_1\alpha_1}}\right)^{-\frac{1}{2}}\left[\frac{W_0}{S_0}\right]_{\beta_1,0}\cos^{\frac{1}{2}}\psi\,\sin\left(\tfrac{1}{2}\psi+\tfrac{1}{4}\pi\right), \\ {}_s\dot{U}^{(8)}_{\beta_1} &\doteqdot \frac{-4\sqrt{2}\,\beta_1^{\frac{1}{2}}}{x^{\frac{3}{2}}}\left(\frac{2H-z}{\widehat{\beta_1\alpha_1}}\right)^{-\frac{1}{2}}\left[\frac{W_1}{S_0}-\frac{W_0S_1}{S_0^2}-\frac{F_1}{F_0}\frac{W_0}{S_0}\right]_{\beta_1,1}\cos^{\frac{1}{2}}\psi\,\sin\left(\tfrac{1}{2}\psi-\tfrac{1}{4}\pi\right), \\ {}_s\dot{W}^{(8)}_{\beta_1} &\doteqdot \frac{-4\sqrt{2}\,\beta_1^{\frac{1}{2}}h}{\widehat{\beta_1\alpha_1}\,x^{\frac{3}{2}}}\left(\frac{2H-z}{\widehat{\beta_1\alpha_1}}\right)^{-\frac{1}{2}}\left[\frac{W_0}{S_0}\right]_{\beta_1,0}\cos^{\frac{1}{2}}\psi\,\sin\left(\tfrac{1}{2}\psi-\tfrac{1}{4}\pi\right), \\ {}_s\dot{W}^{(8)}_{\beta_1} &\doteqdot \frac{4\sqrt{2}\,\beta_1^{\frac{3}{2}}}{\widehat{\beta_1\alpha_1}\,x^{\frac{3}{2}}}\left(\frac{2H-z}{\widehat{\beta_1\alpha_1}}\right)^{-\frac{1}{2}}\left[\frac{W_1}{S_0}-\frac{W_0S_1}{S_0^2}-\frac{F_1}{F_0}\frac{W_0}{S_0}\right]_{\beta_1,1}\cos^{\frac{1}{2}}\psi\,\sin\left(\tfrac{1}{2}\psi+\tfrac{1}{4}\pi\right), \end{aligned} \right. $$

where $\tau = t-x/\beta_1-h^2/2x\beta_1,\quad p=(2H-z)/\widehat{\beta_1\alpha_1},\quad \tan\psi=\tau/p,$

and similarly

$$ {}_s\phi^{(9)}_{\beta_1} \doteqdot (-)\times\ldots,\quad {}_sU^{(9)}_{\beta_1},\ {}_s\dot{U}^{(9)}_{\beta_1} \doteqdot (-)\times\ldots,\quad {}_sW^{(9)}_{\beta_1},\ {}_s\dot{W}^{(9)}_{\beta_1} \doteqdot (+)\times\ldots $$

with $(2H+z)$ for $(2H-z)$ and $[3F_1W_0/F_0S_0]$ for $[F_1W_0/F_0S_0]$.

${}_s\psi$ *contributions from* Γ_{β_1}

$$ \left\{ \begin{aligned} {}_s\psi^{(1)}_{\beta_1} &\doteqdot 8\sqrt{(2\pi i)}\,\beta_1^{\frac{3}{2}}\widehat{\beta_1\alpha_1}^{-1}x^{-\frac{3}{2}}\{(h+z)\,\omega^{-\frac{1}{2}}+\beta_1[F_1/F_0]_{\beta_1}\omega^{-\frac{3}{2}}\}\exp\{i\omega\tau\}, \\ {}_sU^{(1)}_{\beta_1} &\doteqdot -8\sqrt{2}\,\beta_1^{\frac{1}{2}}\widehat{\beta_1\alpha_1}^{-1}x^{-\frac{5}{2}}(h+z)^2\tau'^{-\frac{1}{2}}H(\tau'), \\ {}_s\dot{U}^{(1)}_{\beta_1} &\doteqdot 8\sqrt{2}\,\beta_1^{\frac{3}{2}}\widehat{\beta_1\alpha_1}^{-1}x^{-\frac{3}{2}}\tau'^{-\frac{1}{2}}H(\tau'), \\ {}_sW^{(1)}_{\beta_1} &\doteqdot 8\sqrt{2}\,\beta_1^{\frac{1}{2}}\widehat{\beta_1\alpha_1}^{-1}x^{-\frac{3}{2}}(h+z)\,\tau'^{-\frac{1}{2}}H(\tau'), \end{aligned} \right. $$

$$\left\{\begin{aligned}
{}_s\psi^{(2)}_{\beta_1} &\doteqdot 16\sqrt{(2\pi i)}\,\beta_1^{\frac{3}{2}}\widehat{\beta_1\alpha_1}^{-1}x^{-\frac{3}{2}}\{(h+z)\,\omega^{-\frac{1}{2}}[W_0/S_0]_{\beta_1}-\beta_1\omega^{-\frac{3}{2}}[(W_1/W_0-S_1/S_0-2F_1/F_0)\,W_0/S_0]_{\beta_1}\}\\
&\qquad\times\exp\{i\omega\tau-\omega p\},\\
{}_s\dot{U}^{(2)}_{\beta_1} &\doteqdot \frac{16\sqrt{2}\,\beta_1^{\frac{3}{2}}}{\widehat{\beta_1\alpha_1}\,x^{\frac{3}{2}}}\left(\frac{2H}{\widehat{\beta_1\alpha_1}}\right)^{-\frac{1}{2}}\left[\frac{W_0}{S_0}\right]_{\beta_1,0}\cos^{\frac{1}{2}}\psi\,\sin\left(\tfrac{1}{2}\psi+\tfrac{1}{4}\pi\right),\\
{}_sW^{(2)}_{\beta_1} &\doteqdot \frac{16\sqrt{2}\,\beta_1^{\frac{1}{2}}(h+z)}{\widehat{\beta_1\alpha_1}\,x^{\frac{3}{2}}}\left(\frac{2H}{\widehat{\beta_1\alpha_1}}\right)^{-\frac{1}{2}}\left[\frac{W_0}{S_0}\right]_{\beta_1,0}\cos^{\frac{1}{2}}\psi\,\sin\left(\tfrac{1}{2}\psi+\tfrac{1}{4}\pi\right),\\
{}_s\dot{W}^{(2)}_{\beta_1} &\doteqdot \frac{16\sqrt{2}\,\beta_1^{\frac{3}{2}}}{\widehat{\beta_1\alpha_1}\,x^{\frac{3}{2}}}\left(\frac{2H}{\widehat{\beta_1\alpha_1}}\right)^{-\frac{1}{2}}\left[\frac{W_1}{S_0}-\frac{W_0S_1}{S_0^2}-\frac{2F_1}{F_0}\frac{W_0}{S_0}\right]_{\beta_1,1}\sin\left(\tfrac{1}{2}\psi-\tfrac{1}{4}\pi\right),
\end{aligned}\right.$$

where $\tau = t-x/\beta_1-(h+z)^2/2x\beta_1,\quad p=(2H/\widehat{\beta_1\alpha_1}),\quad \tan\psi=\tau/p;$

$$\left\{\begin{aligned}
{}_s\psi^{(3)}_{\beta_1} &\doteqdot -4\sqrt{(2\pi i)}\,\beta_1^{\frac{1}{2}}x^{-\frac{3}{2}}\{(H+h-z)\,\omega^{-\frac{1}{2}}[Y_0/S_0]_{\beta_1}-\beta_1\omega^{-\frac{3}{2}}[(Y_1/Y_0-S_1/S_0-F_1/F_0)\,Y_0/S_0]_{\beta_1}\}\\
&\qquad\times\exp\{i\omega\tau-\omega p\},\\
{}_s\dot{U}^{(3)}_{\beta_1} &\doteqdot \frac{4\sqrt{2}\,\beta_1^{\frac{1}{2}}}{x^{\frac{3}{2}}}\left(\frac{H}{\widehat{\beta_1\alpha_1}}\right)^{-\frac{1}{2}}\left[\frac{Y_0}{S_0}\right]_{\beta_1,1}\cos^{\frac{1}{2}}\psi\,\sin\left(\tfrac{1}{2}\psi+\tfrac{1}{4}\pi\right),\\
{}_sW^{(3)}_{\beta_1} &\doteqdot \frac{-4\sqrt{2}\,(H+h-z)}{\beta_1^{\frac{1}{2}}x^{\frac{3}{2}}}\left(\frac{H}{\widehat{\beta_1\alpha_1}}\right)^{-\frac{1}{2}}\left[\frac{Y_0}{S_0}\right]_{\beta_1,1}\cos^{\frac{1}{2}}\psi\,\sin\left(\tfrac{1}{2}\psi+\tfrac{1}{4}\pi\right),\\
{}_sW^{(3)}_{\beta_1} &\doteqdot \frac{-4\sqrt{2}\,\beta_1^{\frac{1}{2}}}{x^{\frac{3}{2}}}\left(\frac{H}{\widehat{\beta_1\alpha_1}}\right)^{-\frac{1}{2}}\left[-\frac{Y_0S_1}{S_0^2}-\frac{F_1}{F_0}\frac{Y_0}{S_0}\right]_{\beta_1,0}\cos^{\frac{1}{2}}\psi\,\sin\left(\tfrac{1}{2}\psi-\tfrac{1}{4}\pi\right),
\end{aligned}\right.$$

where $\tau = t-x/\beta_1-(H+h-z)^2/2x\beta_1,\quad p=H/\widehat{\beta_1\alpha_1},\quad \tan\psi=\tau/p,$

and similarly

$${}_s\psi^{(4)}_{\beta_1}\doteqdot(+1)\times\ldots,\quad {}_s\dot{U}^{(4)}_{\beta_1}\doteqdot(-1)\times\ldots,\quad {}_sW^{(4)}_{\beta_1},\ {}_s\dot{W}^{(4)}_{\beta_1}\doteqdot(+1)\times\ldots$$

with $(H-h+z)$ for $(H+h-z)$,

$${}_s\psi^{(5)}_{\beta_1}\doteqdot(-2)\times\ldots,\quad {}_s\dot{U}^{(5)}_{\beta_1}\doteqdot(+2)\times\ldots,\quad {}_sW^{(5)}_{\beta_1},\ {}_s\dot{W}^{(5)}_{\beta_1}\doteqdot(-2)\times\ldots$$

with $(H+h+z)$ for $(H+h-z)$ and $[-3F_1/F_0]$ for $[-F_1/F_0]$;

$$\left\{\begin{aligned}
{}_s\psi^{(6)}_{\beta_1} &\doteqdot -\sqrt{(2\pi i^3)}\,\beta_1^{\frac{1}{2}}x^{-\frac{1}{2}}\omega^{-\frac{1}{2}}\exp\{i\omega\tau\},\\
{}_s\dot{U}^{(6)}_{\beta_1} &\doteqdot \sqrt{2}\,\beta_1^{-\frac{1}{2}}x^{-\frac{3}{2}}(2H-h-z)\,\tau^{-\frac{1}{2}}H(\tau),\\
{}_sW^{(6)}_{\beta_1} &\doteqdot \sqrt{2}\,\beta_1^{-\frac{1}{2}}x^{-\frac{1}{2}}\tau^{-\frac{1}{2}}H(\tau),
\end{aligned}\right.$$

where $\tau = t-x/\beta_1-(2H-h-z)^2/2x\beta_1,$

and similarly

$${}_s\psi^{(7)}_{\beta_1}\doteqdot(-1)\times\ldots,\quad {}_sU^{(7)}_{\beta_1}\doteqdot(-1)\times\ldots,\quad {}_sW^{(7)}_{\beta_1}\doteqdot(-1)\times\ldots$$

with $(2H+h-z)$ for $(2H-h-z)$,

$${}_s\psi^{(8)}_{\beta_1}\doteqdot(-1)\times\ldots,\quad {}_sU^{(8)}_{\beta_1}\doteqdot(+1)\times\ldots,\quad {}_sW^{(8)}_{\beta_1}\doteqdot(-1)\times\ldots$$

with $(2H-h+z)$ for $(2H-h-z)$,

$${}_s\psi^{(9)}_{\beta_1}\doteqdot(+1)\times\ldots,\quad {}_sU^{(9)}_{\beta_1}\doteqdot(-1)\times\ldots,\quad {}_sW^{(9)}_{\beta_1}\doteqdot(+1)\times\ldots$$

with $(2H+h+z)$ for $(2H-h-z)$.

Modifications for $z = 0$

${}_s\phi$ contributions from Γ_{β_1}

$$\begin{cases} {}_sU^{(1)}_{\beta_1} \doteqdot 4\sqrt{2}\beta_1^{-\frac{1}{2}}x^{-\frac{3}{2}}h\tau^{\frac{1}{2}}H(\tau), \\ {}_s\dot{U}^{(1)}_{\beta_1} \doteqdot -4\sqrt{2}\beta_1^{\frac{1}{2}}x^{-\frac{3}{2}}[F_1/F_0]_{\beta_1,1}\tau'^{-\frac{1}{2}}H(\tau'), \end{cases}$$

$$\begin{cases} {}_sW^{(1)}_{\beta_1} \doteqdot -4\sqrt{2}\beta_1^{\frac{1}{2}}\widehat{\beta_1\alpha_1}^{-1}x^{-\frac{3}{2}}h\tau'^{\frac{1}{2}}H(\tau'), \\ {}_s\dot{W}^{(1)}_{\beta_1} \doteqdot 8\sqrt{2}\beta_1^{\frac{3}{2}}\widehat{\beta_1\alpha_1}^{-1}x^{-\frac{3}{2}}[F_1/F_0]_{\beta_1,1}\tau^{-\frac{1}{2}}H(\tau), \end{cases}$$

where $$\tau = -\tau' = t - x/\beta_1 - h^2/2x\beta_1,$$

and similarly

$${}_sU^{(2)}_{\beta_1},\ {}_s\dot{U}^{(2)}_{\beta_1} \doteqdot (-1)\times\ldots,\quad {}_sW^{(2)}_{\beta_1},\ {}_s\dot{W}^{(2)}_{\beta_1} \doteqdot (-1)\times\ldots$$

with $(2H-h)$ for h and $[2S_1/S_0+F_1/F_0]$ for $[F_1/F_0]$,

$${}_sU^{(3)}_{\beta_1},\ {}_s\dot{U}^{(3)}_{\beta_1} \doteqdot (+1)\times\ldots,\quad {}_sW^{(3)}_{\beta_1},\ {}_s\dot{W}^{(3)}_{\beta_1} \doteqdot (+1)\times\ldots$$

with $(2H+h)$ for h and $[2S_1/S_0+3F_1/F_0]$ for $[F_1/F_0]$.

${}_s\phi$ contributions from Γ_{β_2}

$$\begin{cases} {}_sU^{(2)}_{\beta_2} \doteqdot 8\sqrt{2}\,\mathrm{i}\,\beta_2^{\frac{4}{2}}(2/\beta_2^2-1/\beta_1^2)\,[x\beta_2-(2H-h)\,\widehat{\beta_1\beta_2}]^{-\frac{3}{2}}\,[(T_1/S_0-T_0S_1/S_0^2)/F_0]_{\beta_2,0}\,\tau^{\frac{1}{2}}H(\tau), \\ {}_s\dot{U}^{(2)}_{\beta_2} \doteqdot 4\sqrt{2}\beta_2^{\frac{4}{2}}(2/\beta_2^2-1/\beta_1^2)\,[x\beta_2-(2H-h)\,\widehat{\beta_1\beta_2}]^{-\frac{3}{2}}\,[(T_1/S_0-T_0S_1/S_0^2)/F_0]_{\beta_2,1}\,\tau'^{-\frac{1}{2}}H(\tau'), \end{cases}$$

$$\begin{cases} {}_sW^{(2)}_{\beta_2} \doteqdot 8\sqrt{2}\beta_2^{\frac{5}{2}}\widehat{\beta_2\alpha_1}^{-1}(2/\beta_2^2-1/\beta_1^2)\,[x\beta_2-(2H-h)\,\widehat{\beta_1\beta_2}]^{-\frac{3}{2}}\,[(T_1/S_0-T_0S_1/S_0^2)/F_0]_{\beta_2,1}\,\tau^{\frac{1}{2}}H(\tau), \\ {}_s\dot{W}^{(2)}_{\beta_2} \doteqdot -4\sqrt{2}\,\mathrm{i}\,\beta_2^{\frac{5}{2}}\widehat{\beta_2\alpha_1}^{-1}(2/\beta_2^2-1/\beta_1^2)\,[x\beta_2-(2H-h)\,\widehat{\beta_1\beta_2}]^{-\frac{3}{2}}\,[(T_1/S_0-T_0S_1/S_0^2)/F_0]_{\beta_2,0}\,\tau'^{-\frac{1}{2}}H(\tau'), \end{cases}$$

where $$\tau = -\tau' = t - x/\beta_1 - (2H-h)/\widehat{\beta_1\beta_2},$$

and similarly

$${}_sU^{(3)}_{\beta_1},\ {}_s\dot{U}^{(3)}_{\beta_1} \doteqdot (-1)\times\ldots,\quad {}_sW^{(3)}_{\beta_1},\ {}_sW^{(3)}_{\beta_1} \doteqdot (-1)\times\ldots$$

with $(2H+h)$ for $(2H-h)$ and $[\overline{F}_0/F_0]$ for $[1/F_0]$.

Potentials and displacements contributed by Γ_{γ_1}

${}_p\phi$, ${}_s\psi$ contributions may be written

$$\begin{cases} {}_p\phi_{\gamma_1},\quad {}_s\psi_{\gamma_1} = -2\pi\mathrm{i}[A]_{\gamma_1}\exp\{\mathrm{i}\omega\tau-\omega p\}, \\ \text{with derived displacements} \\ \qquad U_{\gamma_1} = -\dfrac{2\tau[A]_{\gamma_1}}{p^2+\tau^2}, \\ \qquad W_{\gamma_1} = \dfrac{2p[A]_{\gamma_1}}{p^2+\tau^2}, \end{cases}$$

where $[\quad]_{\gamma_1}$ implies evaluation at $\zeta = \kappa_{\gamma_1}$, $\tau = t - x/\gamma_1$, $F' = [\mathrm{d}F/\mathrm{d}\zeta]_{\gamma_1}$.

Contributions from

$$\begin{cases} {}_p\phi^{(1)}_{\gamma_1} & \text{with } A = 8\gamma_1/\widehat{\gamma_1\beta_1}, & p = (h+z)/\widehat{\gamma_1\alpha_1}, \\ {}_p\phi^{(3),(4)}_{\gamma_1} & \text{with } A = -4\widehat{\gamma_1\alpha_1}\beta_1^2(2/\gamma_1^2 - 1/\beta_1^2)\,[Y/S]_{\gamma_1}/\widehat{\beta_1\alpha_1}F', & p = (H\pm h\mp z)/\widehat{\gamma_1\alpha_1} + H/\widehat{\gamma_1\beta_1}, \\ {}_p\phi^{(7),(8)}_{\gamma_1} & \text{with } A = -\widehat{\gamma_1\alpha_1}\,\overline{F}[W/S]_{\gamma_1}/\gamma_1 F', & p = (2H\pm h\mp z)/\widehat{\gamma_1\alpha_1}, \end{cases}$$

$$\begin{cases} {}_s\psi^{(1)}_{\gamma_1} & \text{with } A = 8\gamma_1/\widehat{\gamma_1\alpha_1}F', & p = (h+z)/\widehat{\gamma_1\beta_1}, \\ {}_s\psi^{(3),(4)}_{\gamma_1} & \text{with } A = -4\widehat{\gamma_1\beta_1}\beta_1^2(2/\gamma_1^2 - 1/\beta_1^2)\,[Y/S]_{\gamma_1}/\widehat{\beta_1\alpha_1}F', & p = (H\pm h\mp z)/\widehat{\gamma_1\beta_1} + H/\widehat{\gamma_1\alpha_1}, \\ {}_s\psi^{(7),(8)}_{\gamma_1} & \text{with } A = -\widehat{\gamma_1\beta_1}\,\overline{F}[T/S]_{\gamma_1}/\gamma_1 F', & p = (2H\pm h\mp z)/\widehat{\gamma_1\beta_1}. \end{cases}$$

${}_s\phi$, ${}_p\psi$ contributions may be written

$$\begin{cases} {}_s\phi_{\gamma_1},\quad {}_p\psi_{\gamma_1} = 2\pi[A]_{\gamma_1}\exp\{\mathrm{i}\omega\tau - \omega p\}, \\ \text{with derived displacements} \\ \qquad U_{\gamma_1} = \dfrac{2\tau[A]_{\gamma_1}}{p^2+\tau^2}, \\ \qquad W_{\gamma_1} = -\dfrac{2p[A]_{\gamma_1}}{p^2+\tau^2}, \end{cases}$$

where $[\quad]_{\gamma_1}$ implies evaluation at $\zeta = \kappa_{\gamma_1}$, $\tau = t - x/\gamma_1$, $F' = [\mathrm{d}F/\mathrm{d}\zeta]_{\gamma_1}$.

Contributions from

$$\begin{cases} {}_s\phi^{(1)}_{\gamma_1} & \text{with } A = 4\gamma_1^2(2/\gamma_1^2 - 1/\beta_1^2)/F', & p = z/\widehat{\gamma_1\alpha_1} + h/\widehat{\gamma_1\beta_1}, \\ {}_s\phi^{(2)}_{\gamma_1} & \text{with } A = -4(2/\gamma_1^2 - 1/\beta_1^2)\,[T/S]\,\gamma_1^2/F', & p = z/\widehat{\gamma_1\alpha_1} + (2H-h)/\widehat{\gamma_1\beta_1}, \\ {}_s\phi^{(5),(6)}_{\gamma_1} & \text{with } A = -\widehat{\gamma_1\alpha_1}\,\overline{F}[Y/S]_{\gamma_1}/\gamma_1 F', & p = (H\mp z)/\widehat{\gamma_1\alpha_1} + (H\pm h)/\widehat{\gamma_1\beta_1}, \\ {}_s\phi^{(8)}_{\gamma_1} & \text{with } A = -4(2/\gamma_1^2 - 1/\beta_1^2)\,\gamma_1^2\,[W/S]_{\gamma_1}/F', & p = (2H-z)/\widehat{\gamma_1\alpha_1} + h/\widehat{\gamma_1\beta_1}, \end{cases}$$

$$\begin{cases} {}_p\psi^{(1)}_{\gamma_1} & \text{with } A = -4\gamma_1^2(2/\gamma_1^2 - 1/\beta_1^2)/F', & p = z/\widehat{\gamma_1\beta_1} + h/\widehat{\gamma_1\alpha_1}, \\ {}_p\psi^{(2)}_{\gamma_1} & \text{with } A = 4\gamma_1^2(2/\gamma_1^2 - 1/\beta_1^2)\,[T/S], & p = h/\widehat{\gamma_1\alpha_1} + (2H-z)/\widehat{\gamma_1\beta_1}, \\ {}_p\psi^{(5),(6)}_{\gamma_1} & \text{with } A = \widehat{\gamma_1\alpha_1}\,\overline{F}[Y/S]_{\gamma_1}/\gamma_1 F', & p = (H\mp h)/\widehat{\gamma_1\alpha_1} + (H\pm z)/\widehat{\gamma_1\beta_1}, \\ {}_p\psi^{(8)}_{\gamma_1} & \text{with } A = 4(2/\gamma_1^2 - 1/\beta_1^2)\,\gamma_1^2[W/S]_{\gamma_1}/F', & p = (2H-h)/\widehat{\gamma_1\alpha_1} - z/\widehat{\gamma_1\beta_1}. \end{cases}$$

References

Birch, F. & Law, R. 1935 *Bull. Geol. Soc. Amer.* **46**, 1219–1250.
Bromwich, T. J. I'A. 1898 *Proc. Lond. Math. Soc.* (1) **30**, 98–120.
Bromwich, T. J. I'A. 1916 *Proc. Lond. Math. Soc.* (2) **15**, 425–430.
Copson, E. T. 1935 *Functions of a complex variable.* Oxford: Clarendon Press.
Jeffreys, H. 1926*a* *Proc. Camb. Phil. Soc.* **23**, 472–481.
Jeffreys, H. 1926*b* *Mon. Not. R. Astr. Soc.*, Geophys. Suppl., **1**, 282–293.

MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES
PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A

Jeffreys, H. 1929 *The Earth*, 2nd ed. Cambridge University Press.
Jeffreys, H. 1931 *Beitr. Geophys.* **30**, 336–350.
Jeffreys, H. 1935 *Mon. Not. R. Astr. Soc.*, Geophys. Suppl., **3**, 257.
Jeffreys, H. 1946 *Rep. Progr. Phys.* **10**, 52–82.
Jeffreys, H. & Jeffreys, B. S. 1946 *Methods of mathematical physics*. Cambridge University Press.
Lamb, H. 1904 *Phil. Trans.* A, **203**, 1–42.
Lapwood, E. R. 1949 *Phil. Trans.* A, **242**, 63–100.
Lee, A. W. 1935 See Jeffreys 1935, p. 259.
Longuet-Higgins, M. S. 1950 *Phil. Trans.* A, **243**, 1–35.
Love, A. E. H. 1906 *Mathematical theory of elasticity*, 2nd ed. Cambridge University Press.
Love, A. E. H. 1911 *Some problems in geodynamics*. Cambridge University Press.
Muskat, M. 1933 *Physics*, **4**, 14–28.
Nakano, H. 1925 *Jap. J. Astr. Geophys.*, **2**, 233–326.
Newlands, M. 1950 *Mon. Not. R. Astr. Soc.*, Geophys. Suppl., **6**, 109–124.
Pekeris, C. L. 1935 *Physics*, **6**, 133.
Pekeris, C. L. 1948 *Geol. Soc. Amer.* Mem. no. 27.
Press, F., Ewing, M. & Tolstoy, I. 1950 *Bull. Seis. Soc. Amer.* **40**, 111–148.
Lord Rayleigh (Strutt, J. W.) 1885 *Proc. Lond. Math. Soc.* **17**, 4–11.
Scholte, J. C. 1942 *Proc. Acad. Sci. Amst.* **45**, 20–25, 159–164.
Sommerfeld, A. 1909 *Ann. Phys., Lpz.*, **28**, 665–736.
Stewart, C. A. 1940 *Advanced calculus*. London: Methuen.
Stoneley, R. 1924 *Proc. Roy. Soc.* A, **106**, 416–428.
Stoneley, R. 1934 *Mon. Not. R. Astr. Soc.*, Geophys. Suppl., **3**, 222–232.
Stoneley, R. 1937 *Mon. Not. R. Astr. Soc.*, Geophys. Suppl., **4**, 43–50 and papers there cited.
Watson, G. N. 1922 *Bessel functions*, 2nd ed. Cambridge University Press.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES